Negation Introduction allows us to derive the negation of a sentence if it leads to a contradiction. If we believe $(\varphi \Rightarrow \psi)$ and $(\varphi \Rightarrow \neg\psi)$, then we can derive that φ is false. *Negation Elimination* allows us to delete double negatives.

Negation Introduction

$$\frac{\begin{array}{l}\varphi \Rightarrow \psi \\ \varphi \Rightarrow \neg\psi\end{array}}{\neg\varphi}$$

Negation Elimination

$$\frac{\neg\neg\varphi}{\varphi}$$

Implication Introduction is the structured rule we saw in section 4.3. If, by assuming φ, we can derive ψ, then we can derive $(\varphi \Rightarrow \psi)$. *Implication Elimination* is the first rule we saw Section 4.2.

Implication Introduction

$$\frac{\varphi \vdash \psi}{\varphi \Rightarrow \psi}$$

Implication Elimination

$$\frac{\begin{array}{l}\varphi \Rightarrow \psi \\ \varphi\end{array}}{\psi}$$

Biconditional Introduction allows us to deduce a biconditional from an implication and its inverse.*Biconditional Elimination* goes the other way, allowing us to deduce two implications from a single biconditional.

Biconditional Introduction

$$\frac{\begin{array}{l}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{array}}{\varphi \Leftrightarrow \psi}$$

Biconditional Elimination

$$\frac{\varphi \Leftrightarrow \psi}{\begin{array}{l}\varphi \Rightarrow \psi \\ \psi \Rightarrow \varphi\end{array}}$$

In addition to these rules of inference, it is common to include in Fitch proof editors several additional operations that are of use in constructing Fitch proofs. For example, the Premise operation allows one to add a new premise to a proof. The Reiteration operation allows one to reproduce an earlier conclusion for the purposes of clarity. Finally, the Delete operation allows one to delete unnecessary lines.

Unsatisfiability Theorem

$\Delta \models \phi$ iff $\Delta \cup \neg\phi$ is unsatisfiable

Every truth assignment that satisfies Δ must fail to satisfy $\neg\phi$ which means it must satisfy ϕ

$\therefore \Delta \models \phi$

Deduction Theorem

$\phi \models \psi$ iff $\phi \rightarrow \psi$ is valid

Consistency theorm

ϕ is consistent with ψ iff

$\phi \wedge \psi$ is satisfiable

4.5 Reasoning Tips

The Fitch rules are all fairly simple to use; and, as we discuss in the next section, they are all that we need to prove any result that follows logically from any set of premises. Unfortunately, figuring out which rules to use in any given situation is not always that simple. Fortunately, there are a few tricks that help in many cases.

If the goal has the form $(\varphi \Rightarrow \psi)$, it is often good to assume φ and prove ψ and then use Implication Introduction to derive the goal. For example, if we have a premise q and we want to prove $(p \Rightarrow q)$, we assume p, reiterate q, and then use Implication Introduction to derive the goal.

1.	q	Premise
2.	p	Assumption
3.	q	Reiteration: 1
4.	$p \Rightarrow q$	Implication Introduction: 2, 3

If the goal has the form $(\varphi \land \psi)$, we first prove φ and then prove ψ and then use And Introduction to derive $(\varphi \land \psi)$.

If the goal has the form $(\varphi \lor \psi)$, all we need to do is to prove φ or prove ψ, but we do not need to prove both. Once we have proved either one, we can disjoin that with anything else whatsoever.

If the goal has the form $(\neg\varphi)$, it is often useful to assume φ and prove a contradiction, meaning that φ must be false. To do this, we assume φ and derive some sentence ψ leading to $(\varphi \Rightarrow \psi)$. We assume φ again and derive some sentence $\neg\psi$ leading to $(\varphi \Rightarrow \neg\psi)$. Finally, we use Negation Introduction to derive $\neg\varphi$ as desired.

More generally, whenever we want to prove a sentence φ of any sort, we can sometimes succeed by assuming $\neg\varphi$, proving a contradiction as just discussed and thereby deriving $\neg\neg\varphi$. We can then apply Negation Elimination to get φ.

The following two tips suggest useful things we can try based on the form of the premises and the goal or subgoal we are trying to prove.

If there is a premise of the form $(\varphi \Rightarrow \psi)$ and our goal is to prove ψ, then it is often useful to try proving φ. If we succeed, we can then use Implication Elimination to derive ψ.

$$\frac{\neg(\neg p)}{p}$$

$$\frac{\begin{array}{l} p \to q \\ q \to r \end{array}}{}$$

assume p

q I E

prove

$p \to r$

If we have a premise $(\varphi \vee \psi)$ and our goal is to prove χ, then we should try proving $(\varphi \Rightarrow \chi)$ and $(\psi \Rightarrow \chi)$. If we succeed, we can then use Or Elimination to derive χ.

As an example of using these tips in constructing the proof, consider the following problem. We are given $p \vee q$ and $\neg p$, and we are asked to prove q. Since the goal is not an implication or a conjunction or a disjunction or a negation, only the last of the goal-based tips applies. Unfortunately, this does not help us in this case. Luckily, the second of the premise-based tips is relevant because we have a disjunction as a premise. To use this all we need is to prove $p \Rightarrow q$ and $q \Rightarrow q$. To prove $p \Rightarrow q$, we use the first goal-based tip. We assume p and try to prove q. To do this we use that last goal-based tip. We assume $\sim q$ and prove p. Then we assume $\sim q$ and prove $\neg p$. Since we have proved p and $\neg p$ from $\neg q$, we can infer q. Using Implication Introduction, we then have $p \Rightarrow q$. Proving $q \Rightarrow q$ is easy. Finally, we can apply or elimination to get the desired result.

1.	$p \mid q$	Premise
2.	$\neg p$	Premise
3.	p	Assumption
4.	$\neg q$	Assumption
5.	p	Reiteration: 3
6.	$\neg q \Rightarrow p$	Implication Introduction: 4, 5
7.	$\neg q$	Assumption
8.	$\neg p$	Reiteration: 2
9.	$\neg q \Rightarrow \neg p$	Implication Introduction: 7, 8
10.	$\neg\neg q$	Negation Introduction: 6, 9
11.	q	Negation Elimination: 10
12.	$p \Rightarrow q$	Implication Introduction: 3, 11
13.	q	Assumption
14.	$q \Rightarrow q$	Implication Introduction: 13
15.	q	Or Elimination: 1, 12, 14

In general, when trying to generate a proof, it is useful to apply the premise tips to derive conclusions. However, this often works only for very short proofs. For more complex proofs, it is

Valid — $p \vee \neg p$ } satisfiable

contingent

unsatisfiable } falsifiable

example validity $(p \to q) \vee (q \to r)$

$p \to (q \to p)$

p	q	$q \to p$	$p \to (q \to p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

$p \Leftrightarrow \neg\neg p$

Logically Equivalent

de Morgans Laws
$$\boxed{\begin{array}{l} \neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q) \\ \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \end{array}}$$

implication introduction

$p \to (q \to p)$

implication distribution

$p \to (q \to r) \to (p \to q) \to (p \to r)$

Logical Entailment

$\Delta \models \phi$ Δ logically entails ϕ iff every interpretation that satisfies the premises also satisfies the conclusions.

$\{p\} \models \{p, q\}$ $\{p\} \models p \vee q$ $p \models p, q$

Logical Entailment $\neq$ Logical Equivalence

consider $\{p\} \models p \vee q$ but $\{p \vee q\} \not\models \{p\}$

often useful to think backwards from the desired conclusion before starting to prove things from the premises in order to devise a strategy for approaching the proof. This often suggests subproblems to be solved. We can then work on these simpler subproblems and put the solutions together to produce a proofs for our overall conclusion.

4.6 Soundness And Completeness

In talking about Logic, we now have two notions - logical entailment and provability. A set of premises logically entails a conclusion if and only if every truth assignment that satisfies the premises also satisfies the conclusion. A sentence is provable from a set of premises if and only if there is a finite proof of the conclusion from the premises.

The concepts are quite different. One is based on truth assignments; the other is based on symbolic manipulation of expressions. Yet, for the proof systems we have been examining, they are closely related.

We say that a proof system is *sound* if and only if every provable conclusion is logically entailed. In other words, if $\Delta \vdash \varphi$, then $\Delta \models \varphi$. We say that a proof system is *complete* if and only if every logical conclusion is provable. In other words, if $\Delta \models \varphi$, then $\Delta \vdash \varphi$.

The Fitch system is sound and complete for the full language. In other words, for this system, logical entailment and provability are identical. An arbitrary set of sentences Δ logically entails an arbitrary sentence φ if and only if φ is provable from Δ using Fitch.

The upshot of this result is significant. On large problems, the proof method often takes fewer steps than the truth table method. (Disclaimer: In the worst case, the proof method may take just as many or more steps to find an answer as the truth table method.) Moreover, proofs are usually much smaller than the corresponding truth tables. So writing an argument to convince others does not take as much space.

Recap

A *pattern* is an expression satisfying the grammatical rules of our language except for the occurrence of *metavariables* in place of various subparts of the expression. An *instance* of a pattern is the expression obtained by substituting expressions of the appropriate sort for the metavariables in the pattern so that the result is a legal expression. A *rule of inference* is a pattern of reasoning consisting of one set of patterns, called *premises*, and a second set of schemas, called *conclusions*. A *linear proof* of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either (1) a premise or (2) the result of applying a rule of inference to earlier items in sequence. If there exists a proof of a sentence φ from a set Δ of premises and the axiom schemas and rules of inference of a proof system, then φ is said to be *provable* from Δ (written as $\Delta \vdash \varphi$) and is called a *theorem* of Δ. *Fitch* is a powerful yet

~~# of heads = 60~~
~~# of~~

Choose 60 coins

his # of heads = x

his # of tails = $60 - x$

Flip them all

There were 60 Heads to begin with

Sister has $60 - x$ heads

Does p logically entail $p \vee q$

use truth table

$\{p \to q,\ m \to p \vee q,\ m\} \models q$

m	p	q	$p \vee q$	$m \to (p \vee q)$	$p \to q$	m	q
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	F	T
F	T	F	T	T	F	F	F
F	F	T	T	T	T	F	T
F	F	F	F	T	T	F	F

simple proof system that supports structured proofs. A proof system is *sound* if and only if every provable conclusion is logically entailed. A proof system is *complete* if and only if every logical conclusion is provable. Fitch is sound and complete for Propositional Logic.

Exercises

Exercise 4.1: Given p and q and $(p \wedge q \Rightarrow r)$, use the Fitch system to prove r.

Exercise 4.2: Given $(p \wedge q)$, use the Fitch system to prove $(q \vee r)$.

Exercise 4.3: Given $p \Rightarrow q$ and $q \Leftrightarrow r$, use the Fitch system to prove $p \Rightarrow r$.

Exercise 4.4: Given $p \Rightarrow q$ and $m \Rightarrow p \vee q$, use the Fitch System to prove $m \Rightarrow q$.

Exercise 4.5: Given $p \Rightarrow (q \Rightarrow r)$, use the Fitch System to prove $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$.

Exercise 4.6: Use the Fitch System to prove $p \Rightarrow (q \Rightarrow p)$.

Exercise 4.7: Use the Fitch System to prove $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

Exercise 4.8: Use the Fitch System to prove $(\neg p \Rightarrow q) \Rightarrow ((\neg p \Rightarrow \neg q) \Rightarrow p)$.

Exercise 4.9: Given p, use the Fitch System to prove $\neg\neg p$.

Exercise 4.10: Given $p \Rightarrow q$, use the Fitch System to prove $\neg q \Rightarrow \neg p$.

Exercise 4.11: Given $p \Rightarrow q$, use the Fitch System to prove $\neg p \vee q$.

Exercise 4.12: Use the Fitch System to prove $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$.

Exercise 4.13: Given $\neg(p \vee q)$, use the Fitch system to prove $(\neg p \wedge \neg q)$.

Exercise 4.14: Use the Fitch system to prove the tautology $(p \vee \neg p)$.

1.
2
5
10

10 + 1
5 + 1
11

10 5

1 + 2 2
1 3
1 + 5 8
1 9
1 + 10 19

1 + 2 2
1 3
10 + 5 10
2 12
2 + 1 14

10 + 5 18
5 15

10 + 2 10
2 12

Literal – atomic sentence or negation of atomic

clausal sentence : a literal or disjunctin of literals.

p	$\{p\}$
$\neg p$	$\{\neg p\}$
$\neg p \vee p$	$\{\neg p, p\}$

Conversions

Implications

$\phi \to \psi \longrightarrow \neg\phi \vee \psi \longrightarrow \{\neg\phi, \psi\}$

$\phi \leftarrow \psi \longrightarrow \phi \vee \neg\psi \longrightarrow \{\phi, \neg\psi\}$

Negations

$\neg\neg\phi \longrightarrow \phi \longrightarrow \{\phi\}$

$\neg(\phi \wedge \psi) \to \neg\phi \vee \neg\psi \to \{\neg\phi, \neg\psi\}$

$\neg(\phi \vee \psi) \to \neg\phi \wedge \neg\psi \to \{\neg\phi\}, \{\neg\psi\}$

Distributin

$$\phi \vee (\psi \wedge \chi) \to (\phi \vee \psi) \wedge (\phi \vee \chi)$$

$$(\phi \wedge \psi) \vee \chi \to (\phi \vee \chi) \wedge (\psi \vee \chi)$$

$$\phi \vee (\phi_1 \vee \dots \vee \phi_n) \to \phi \vee \phi_1 \dots \vee \phi_n$$

$$(\phi_1 \vee \dots \vee \phi_n) \vee \phi \to \phi_1 \vee \dots \vee \phi_n \vee \phi$$

$$\phi \wedge (\phi_1 \wedge \dots \wedge \phi_n) \to \phi_1 \wedge \dots \wedge \phi_n \wedge \phi$$

Operators

$$\phi_1 \vee \dots \vee \phi_n \to \{\phi_1, \dots, \phi_n\}$$

$$\phi_1 \wedge \dots \wedge \phi_n \to \{\phi_1\}, \dots, \{\phi_n\}$$

③

convert $\neg(g \wedge (r \rightarrow f))$

I
Implications

$\neg(g \wedge (\neg r \vee f))$

N
egations

$\neg g \vee \neg(\neg r \vee f)$

$\neg g \vee (\neg\neg r \wedge \neg f)$

D
istributions

$(\neg g \vee \neg\neg r) \wedge (\neg g \vee \neg f)$

O
perators

$\{\neg g, r\}, \{\neg g, \neg f\}$

(4)

Convert $g \wedge (r \Rightarrow f)$

Implications $g \wedge (\neg r \vee f)$

Negations $g \wedge (\neg r \vee f)$

Distributions $\cancel{(g \wedge \neg r) \vee (g \wedge f)}$

Operators $\cancel{\{g\},\{\neg r, f\}}$

$\{g\}, \{\neg r, f\}$

Resolution

Suppose we have

$\{p, q\}$ here we know $p=T$ or $q=T$

$\{\neg p, r\}$ here we know $p=F$ or $r=T$

Since we know $\underbrace{p=T}_{\text{case1}}$ or $\underbrace{p=F}_{\text{case2}}$ $(\neg p=T)$

then we know

case 1 $p=T$ means $\neg p=F$ so $r=T$

case 2 $p=F$ means $q=T$

$\{q, r\}$

$$\frac{\{\phi_1, \ldots, \chi, \ldots, \phi_m\} \qquad \{\psi_1, \ldots, \neg\chi, \ldots, \psi_n\}}{\{\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n\}}$$

$$\frac{\{p, q\} \qquad \{\neg p, r\}}{\{q, r\}}$$

Clauses are sets, so we do not have two occurances of the same literal in one set

$$\begin{array}{c}\{\neg p, r\}\\ \{p, r\}\\ \hline \{r\}\end{array}$$

example

$$\begin{array}{c}\{p, q, r\}\\ \{\neg p\}\\ \hline \{q, r\}\end{array}$$

example

$$\begin{array}{c}\{\neg p\}\\ \{p\}\\ \hline \{\}\end{array}$$

NTS

$\Rightarrow p \wedge \neg p$

No members in that set

$$\frac{\begin{array}{c}\{p, q\} \\ \{\neg p, \neg q\}\end{array}}{\begin{array}{c}\{p, \neg p\} \\ \{q, \neg q\}\end{array}}$$

If two clauses have multiple pairs of complementary literals ONLY RESOLVE ONE at a TIME

$$\frac{\begin{array}{c}\{p, q\} \\ \{\neg p, \neg q\}\end{array}}{\{\ \}}$$ WRONG!

This would imply the two sets or clauses are inconsistent. But, they are <u>NOT</u> inconsistent

i.e. $p = T \quad q = F$

Satisfies bothe clauses

Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

=T

$p \to q$

propositional rule of inference

$$\frac{\begin{array}{c} p \to q \\ p \end{array}}{q}$$

$\{\neg p, q\}$ this clause is true if $p = F$ or $q = T$

$\{p\}$ this clause is true $p = T$

$\{q\}$ this clause is true if $q = T$

linear proof

goal $m \to q$

1. $m \to p \vee q$
2. $p \to q$

---

3 | m assumption
4. | $p \vee q$ IE(1,3)
5. | | q assumption
6. | | q reiteration (5)
7. | $q \to q$ II (5,6)
8. | $p \to q$ reiteration (2)
9. | q OE (4,7,8)

10 $m \to q$ II (3, 9)

Resolution ~~Proof~~ derivation

10

$\{\neg m, p, q\}$

$\{\neg p, q\}$

$\{\neg m, q\}$

$m \to q$

Example of Resolution Derivation

1. $\{\neg p, r\}$ premise $p \to r$
2. $\{\neg q, r\}$ premise $q \to r$
3. $\{p, q\}$ premise $\neg p \to q$

---

4. $\{q, r\}$ 1,3
5. $\{r\}$ 2,4

---

note resolution is <u>NOT</u> generatively complete.
i.e cannot derive all clauses logically entailed by the premises.

$\{p\}$
$\{q\}$

---

no resolution

But

p
q } these premises

$p \vee q$ } entail this result

Consider a set of sentences Δ (12)

$$\left.\begin{array}{ll} 1 & \{p, q\} \\ 2 & \{p, \neg q\} \\ 3 & \{\neg p, q\} \\ 4 & \{\neg p, \neg q\} \end{array}\right\} \quad \Delta \text{ is a set of premises}$$

---

5. $\{p\}$ 1,2
6. $\{\neg p\}$ 3,4
7. $\{\ \}$ 5,6

~~Δ⊨~~

How can we determine entailment?

From Unsatisfiability theorem

recall a set of sentences Δ logically entails a sentence ϕ iff

$\Delta \cup \neg\phi$ is inconsistent

So, to determine logical entailment

STEP 1 Negate Goal $\neg\phi$

STEP 2 Add it to premises $\Delta \cup \neg\phi$

STEP 3 Use resolution to determine if $\Delta \cup \neg\phi$ is unsatisfiable that is results in the empty set.

resolution proof

A sentence ϕ is provable from a set of premises Δ

that is $\Delta \vdash \phi$

IFF

$\Delta \cup \neg\phi$ resolves to $\{\}$

OR.

A resolution proof of ϕ from Δ is a resolution derivation of the empty set $\{\}$ from

$\Delta \cup \neg\phi$

Example

1 $\{p\}$

2 $\{\neg p, q\}$

3 $\{p, \neg q, r\}$

4 $\{\neg q, r\}$

5 $\{\neg r\}$

---

6. $\{q\}$ 1, 2

7. $\{r\}$ 4, 6

8 $\{\}$ 5, 7 which proves

$$\begin{array}{l} p \\ p \to q \\ p \vee q \to r \\ q \to r \\ \hline r \end{array}$$

linear proof

goal r

(16)

1. p premise
2. $p \to q$ premise
3. $(p \to q) \to (q \to r)$ premise

---

4. $q \to r$ IE (2,3)
5. q IE (1,2)
6. r IE (4,5)

convert

$(p \to q) \to (q \to r)$

$(\neg p \vee q) \to (\neg q \vee r)$

$\neg(\neg p \vee q) \vee (\neg q \vee r)$

$(p \wedge \neg q) \vee (\neg q \vee r)$

$(p \vee \neg q \vee r) \wedge (\neg q \vee \neg q \vee r)$

$\{p, \neg q, r\}, \{\neg q, r\}$

resolution proof

1. $\{p\}$ premise
2. $\{\neg p, q\}$ premise
3. $\{p, \neg q, r\}$ premise
4. $\{\neg q, r\}$ premise
5. $\{\neg r\}$ negation of goal

---

6. $\{q\}$ 1,2
7. $\{r\}$ 4,6
8. $\{\}$ 5,7

using resolution to prove validity

no premises prove $(p \to (q \to p))$

negate the goal

$\neg(p \to (q \to p))$

$\neg(p \to (\neg q \vee p))$

$\neg(\neg p \vee (\neg q \vee p))$

$\neg\neg p \wedge \neg(\neg q \vee p)$

$p \wedge q \wedge \neg p$

1 $\{p\}$

2 $\{q\}$

3 $\{\neg p\}$

---

4 $\{\}$ 1,3