Phil. 225 - Symbolic Logic
HW 11 - Possible answers

1) a) Let $B_1, \dots B_{n+1}$ be a proof, all of whose premises are included in $\{A_1, \dots A_k\}$, where $B_n = C$ and $B_{n+1} = (C \vee D)$ [and so comes from line n by DI]. Assume (IH) that every line $\leq n$ is a consequence of its premises. Then specifically, $\{A_1, \cdots A_k\} \vDash C$. Show $\{A_1, \cdots A_k\} \vDash (C \vee D)$. Let $\mathcal{M}$ be any structure in which all of $A_1 \cdots A_k$ are true. Then $\overset{m}{\vDash} C$. But by truth def. of '$\vee$', $\overset{m}{\vDash}(C \vee D)$. Hence $\{A_1 \dots A_k\} \vDash (C \vee D$. Q.E.D.

b) Let $B_1 \dots B_{n+1}$ be a proof, all of whose premises are included in $\{A_1 \cdots A_k\}$, where $B_n = \forall x Ax$ and $B_{n+1} = A(c)$. [So $B_{n+1}$ comes from $B_n$ by $\forall$-elim] Here 'c' is any constant. Assume (IH) that $\{A_1 \cdots A_k\} \vDash \forall x Ax$. Show $\{A_1 \cdots A_k\} \vDash A(c)$. Let $\mathcal{M}$ be any structure in which $A_i$ are all true, $1 \leq i \leq k$. Show $\overset{m}{\vDash} A(c)$. Since $\{A_1 \dots A_k\} \vDash \forall x Ax$ and all $A_i$ are true in $\mathcal{M}$, $\overset{m}{\vDash} \forall x Ax$. By def of satisfaction for $\forall$, every element of $D_m$ satisfies $P(x)$ in $\mathcal{M}$. Specifically, let $d = c^m$ (i.e., $d$ is the element of $D_m$ that $\mathcal{M}$ assigns to the constant 'c'). Then $\overset{m}{\vDash} P(x)\, g[x/d]$. But $\overset{m}{\vDash} P(x)[g\, x/d]$ iff $\overset{m}{\vDash} P(c)$, since $g(x/d)$ assigns $d$ to 'x' and $\mathcal{M}$ assigns $d$ to 'c'. Hence $\overset{m}{\vDash} P(c)$. Q.E.D.

2) $\overline{\overline{\mathbb{N}}} = \aleph_0$, so by def. of multiplication, it will suffice to show that $\overline{\overline{\mathbb{N} \times \mathbb{N}}} = \aleph_0$. List all ordered pairs of natural numbers according to the sum of the pairs (and for pairs of equal sums, lexically with the smaller 1st element preceding). Thus:
$\langle 0,0 \rangle, \langle 0,1 \rangle, \langle 1,0 \rangle, \langle 0,2 \rangle, \langle 1,1 \rangle, \langle 2,0 \rangle, \langle 0,3 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,0 \rangle, \dots$
This establishes a 1-1 correspondence between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.
So $\overline{\overline{\mathbb{N} \times \mathbb{N}}} = \aleph_0$.

3) Each point in 3-dimensional space can be represented uniquely as an ordered triple of real numbers $t = \langle r_1, r_2, r_3 \rangle$. Let each real be represented in decimal notation as $c_1 c_2 \cdots c_n . d_1 d_2 d_3 \cdots\cdots$ where all the $c_i$ and $d_i$ are integers between 0 and 9. (Note there are finitely many $c_i$ but $\aleph_0$ $d_i \ldots$) If

$$r_1 = c_1^1 c_2^1 c_3^1 \cdots c_i^1 . d_1^1 d_2^1 d_3^1 \cdots\cdots$$
$$r_2 = c_1^2 c_2^2 c_3^2 \cdots c_j^2 . d_1^2 d_2^2 d_3^2 \cdots\cdots$$
$$r_3 = c_1^3 c_2^3 c_3^3 \cdots c_k^3 . d_1^3 d_2^3 d_3^3 \cdots\cdots$$

, then let $m = \max(i, j, k)$, and let

$$f(r_1, r_2, r_3) = c_1^1 c_1^2 c_1^3 c_2^1 c_2^2 c_2^3 \cdots c_m^1 c_m^2 c_m^3 . d_1^1 d_1^2 d_1^3 d_2^1 d_2^2 d_2^3 d_3^1 d_3^2 d_3^3 \cdots$$

where $c_q^p = 0$ if there was no corresponding integer digit. Thus if

$$r_1 = 123.4567890123\ldots$$
$$r_2 = 45.0101010101\ldots$$
$$r_3 = 6.2222222\ldots$$

then

$$f(r_1, r_2, r_3) = 100240356.402512602712802912002112202\ldots$$

$f$ establishes a 1-1 correspondence between $R^3$ and $R$. (Note that $f$ is <u>ONTO</u>: each real number can be disarticulated into three components.)

One hitch: $.999\overline{9}\ldots = 1.000\overline{0}\ldots$, which will violate $f$ being 1-1. This case will need to be dealt with. (Proof left to the reader!)