Phil 225 - Symbolic Logic
HW10 - possible answers

1. We proved in class that there are $\aleph_0$ positive rationals, so it will suffice to show that there are $\aleph_0$ positive rationals less than 1. Since the positive rationals less than 1 is a proper subset of the positive rationals, there are at most $\aleph_0$ of them. But there are also at least $\aleph_0$ positive rationals less than 1 (consider: ½. 1/3, ¼, 1/5, …). Hence by the Schroeder-Bernstein theorem, there are $\aleph_0$ rationals less than 1.

Here's another way to do it. Given a positive rational number i.abcdef…. , where i is the integer part and abcdef… is the decimal part, first spread out the decimal part by inserting 0s between consecutive digits. (Just like Hotel Infinity.) So .222222…. becomes .202020202020…. and .734000… becomes .7030400000000000.

Now take the integer part and turn it around. So 34 becomes 43, and 3400 becomes 0043. Insert these n turned-around digits in for the first n 0s in the even-numbered slots of the decimal part. So 34.222222…. becomes .4232020202020202…. and 3400.222222….. becomes .0202423202020202…..

Note that decimals that are already between 0 and 1 map into even more spread-out decimals between 0 and 1. So while 34.222222 maps into .423202020202…. , .423202020202… itself maps into .040203020002000200020002000 2….

So the mapping is 1-1 from all positive rationals into all positive rationals between 0 and 1. (Note it is into, not onto. 2/9 = .222222222… is not the image of any number under this mapping.) But since the identity mapping maps the positive rationals between 0 and 1 into all the positive rationals, the into mapping is enough again we're appealing implicitly to the Schroeder-Bernstein theorem).

2a. Let f: $\{a_1, a_2, \ldots a_n\} \rightarrow \{0,1\}$. There are 2 choices (0 or 1) of what a_1 can map into. For each of those two choices, there are 2 independent choices for a_2, for a total of 4 choices. And so on. So for n elements, there are 2• 2• 2•**...** (n-times) choices, hence 2^n possible ways to map $\{a_1, a_2, \ldots a_n\}$ into {0,1}.

2b. We need to show that there are $2^{\aleph_0}$ functions mapping N into {0,1}. It will suffice to establish a 1-1 correspondence between functions mapping N into {0,1} and sets of natural numbers (since we know that there are $2^{\aleph_0}$ such sets).

For each function mapping N into {0,1}, associate the set of all the numbers that it maps into 1. This association is clearly 1-1. Similarly, each set of natural numbers corresponds uniquely to the function that maps just the elements of that set into 1, and all the other numbers into 0. Hence the cardinality of the power set of the natural numbers is the same as the cardinality of the set of functions from natural numbers into {0,1}.