Note on Tools:

Friday, October 14, 2022 1:33 PM

To best emulate the process we're expecting you to go through when working through the quiz, we will use proof editors (Hilbert, Fitch, Robinson) when going through example problems, but we will not use other tools (e.g. Babbage).

Propositional Sentence

Monday, October 17, 2022 2:30 PM

Either a

- **Propositional constant**
  In our case, usually single letters like *p* or *q*

- **Compound expression** using logical operators and parentheses
  ()
  ¬ (negation)
  ∧ (conjunction)
  ∨ (disjunction)
  ⇒ (implication)
  ⇔ (biconditional)

Operator Precedence

Monday, October 17, 2022 2:37 PM

¬ (negation)
∧ (conjunction)
∨ (disjunction)
⇒ (implication)
⇔ (biconditional)

Operator Associativity

Monday, October 17, 2022 2:38 PM

Left-associative:

- $\wedge$ (conjunction)
- $\vee$ (disjunction)

Right-associative:

- $\Rightarrow$ (implication)
- $\Leftrightarrow$ (biconditional)

Precedence (continued)

If surrounded by two occurrences of $\wedge$ or $\vee$, the operand associates with the operator to the left.

$$p \wedge q \wedge r \to ((p \wedge q) \wedge r)$$
$$p \vee q \vee r \to ((p \vee q) \vee r)$$

If surrounded by two occurrences of $\Rightarrow$ or $\Leftrightarrow$, the operand associates with the operator to the right.

$$p \Rightarrow q \Rightarrow r \to (p \Rightarrow (q \Rightarrow r))$$
$$p \Leftrightarrow q \Leftrightarrow r \to (p \Leftrightarrow (q \Leftrightarrow r))$$

Quick notes on Implication

Monday, October 17, 2022 2:48 PM

Antecedent

Consequent

Equivalence:

$$p \Rightarrow q$$

$$\neg p \vee q$$

Key consequence: counterfactuals are true

[Babbage (stanford.edu)]

Interpretation/Truth Assignment

Monday, October 17, 2022 2:58 PM

Propositional interpretation: An association between *propositional constants* and truth values (T/F or 1/0). (Synonym: truth assignment)

$p^i = T$

$q^i = F$

$r^i = T$

Sentential interpretation: An association between *propositional sentences* and truth values.

Note: Each distinct *propositional interpretation* uniquely determines the *sentential interpretation* for a set of sentences containing only those propositions.

$(p)^i \stackrel{?}{=}$

$(q \vee r)^i \stackrel{?}{=}$

$(r \Leftrightarrow \neg p)^i \stackrel{?}{=}$

Satisfaction

Friday, October 14, 2022 12:50 PM

Satisfied: We say a sentence or set of sentences is *satisfied* by a propositional interpretation if-and-only-if the sentential interpretation of the sentence/each sentence in the set is T.

Satisfiable: We say a sentence or set of sentences is *satisfiable* if-and-only-if there exists some interpretation of the propositional constants in the sentence(s) that simultaneously satisfies every sentence.

*** Validity/Contingency/Unsatisfiability ***

Monday, October 17, 2022 3:14 PM

Slides 9 & 10, Lecture 3

Valid: A sentence or set of sentences is *valid* iff every interpretation satisfies it
Contingent: A sentence or set of sentences is *contingent* iff there exists some interpretation that satisfies and some interpretation that falsifies it.
Unsatisfiable: A sentence or set of sentences is *unsatisfiable* iff there does not exist an interpretation that satisfies it.

Satisfiable: Any sentence or set of sentences that is not unsatisfiable.
Falsifiable: Any sentence or set of sentences that is not valid.

Propositional Logic (Practice Question)

Friday, October 14, 2022 12:51 PM

Valid, contingent, or unsatisfiable?

$\neg r \wedge (\neg p \wedge r) \wedge (r \Rightarrow p)$

Valid, contingent, or unsatisfiable?

$((\neg p \vee q) \Rightarrow (p \Rightarrow q)) \wedge q$

Sets of sentences

Monday, October 17, 2022 6:45 PM

We frequently use Delta (Δ) and Gamma (Γ) to denote sets of sentences.

Quick set theory refresher:

- Sets contain elements/members.
- Sets cannot have repeat elements.
- Sets can be empty.
- There is not an order to the elements in a set.
- Some set A is a subset of a set B if-and-only-if every element of A is also an element of B. $A \subseteq B$ $A = \{p\}$ $B = \{p, p \wedge q\}$

Relevant Operations on sets:

- Set intersection: operates on two sets and evaluates to a new set that contains every element that is present in *both of the original sets*. (Never larger than either of the original sets.) $A \cap B$
- Set union: operates on two sets and evaluates to a new set that contains every element that is present in *either of the original sets*. (Never smaller than either of the original sets.) $A \cup B$

Semantics of sets of sentences:
A set of sentences is satisfied if-and-only-if every sentence in the set is true.

(Intuitively, think of it like a conjunction of all of its elements. Different from a conjunction only because sets can have an infinite number of elements, while a conjunction must have a finite number of conjuncts.)

Edge case: the empty set of sentences (denoted {}). $\{\}$
The empty set {} is satisfied.

Why?
There are no sentences, so "every sentence" in the set is true.

Logical Equivalence

Monday, October 17, 2022 6:26 PM

Logical equivalence: Sentences are logically equivalent if-and-only-if they have same value for every propositional interpretation.

Implication:

$p \Rightarrow q$ $\neg p \vee q$

Additional constants:

$p \wedge q$ $p \wedge q \wedge (s \vee \neg s)$

[Babbage (stanford.edu)]

Logical Entailment

Monday, October 17, 2022 6:25 PM

Entailment: A premise (or set of premises) entails a conclusion (or set of conclusions) if-and-only-if every interpretation that satisfies the premise(s) also satisfies the conclusion(s).

Note: entailment is a relationship that can hold between sentences or sets of sentences.
Written out:

$\varphi \models \psi$ (sentence entails sentence)

$\Delta \models \Gamma$ (set entails set)

$\varphi \models \Gamma$ (sentence entails set)

$\Delta \models \psi$ (set entails sentence)

Entailment is not symmetric

If entailment between premises and conclusions is symmetric, then the premises and conclusions are logically equivalent.
"Bidirectional entailment is equivalence"

Important Consequence: Vacuity

Monday, October 17, 2022 6:56 PM

An important consequence of the definition of entailment is **vacuity**.

Vacuity is the principle that unsatisfiable premises entail everything.
I.e. If no interpretation satisfies the premises, then "every interpretation" that satisfies the premises also satisfies the conclusions.

"for loop" analogy:

```
for (interpretation i that satisfies premises) {
    if (i does not satisfy the conclusions) {
        return false;
    }
}
return true;
```

Validity

Monday, October 17, 2022 7:04 PM

If {} entails some sentence φ (or set of sentences Δ), then φ (resp. Δ) is valid.

Why?

- The empty set is satisfied by every interpretation.
- Conclusions are valid if they are satisfied by every interpretation.
- Premises entail conclusions if the conclusions are satisfied by every interpretation that satisfies the premises.

So, if the empty set of premises entails some conclusions, then those conclusions are satisfied by every interpretation. Therefore, the conclusions must be valid.

Monotonicity

Monday, October 17, 2022 7:12 PM

If $\Gamma \models \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \models \varphi$.

"The more you know, the more is entailed."
I.e. adding more premises to a set will **never** cause fewer conclusions to be entailed. However, if you remove premises, then you **may** cause fewer conclusions to be entailed.

Corollary: If $\Delta \not\models \varphi$ and $\Gamma \subseteq \Delta$, then $\Gamma \not\models \varphi$.
"You can't entail more with less knowledge."

Rule of thumb: Monotonicity applies when you take the union of two sets of sentences, but not necessarily when taking their intersection.

$\Delta \models \varphi$

$\Gamma \subseteq \Delta$

$\Gamma \overset{?}{\models} \varphi$

Ramification

Monday, October 17, 2022 7:23 PM

If $\Omega \vDash \Delta$ and $\Gamma \subseteq \Delta$, then $\Omega \vDash \Gamma$.

$\Omega \vDash \Delta$ only if every satisfying interpretation of Ω also satisfies every sentence in Δ. Since $\Gamma \subseteq \Delta$, we know that every sentence in Γ is also a sentence in Δ, which we just saw are all satisfied.

Consistency

Monday, October 17, 2022 7:26 PM

Consistency: A sentence/set of sentences is consistent with another if-and-only-if there exists at least one truth assignment that satisfies both sentences/sets of sentences simultaneously.

Note the lack of "every".
Implicit requirement: There must be a satisfying interpretation for each sentence/set of sentences by itself.

Consequence: unsatisfiable sentences/sets of sentences are inconsistent with everything.

Consistent iff there exists a truth assignment that satisfies both phi and psi

- Existential, so needs at least one SAT truth assignment (cf. UNSAT set being consistent with itself)
- To check: check that the union is not UNSAT (can also check the constituents first)

Bonus: Metatheorem Roundup

Monday, October 17, 2022 7:38 PM

Key takeaway: we can determine many relationships between sentences/sets of sentences by doing a single validity/unsatisfiability check.
This is precisely what the resolution principle can do!
I.e. we can check unsatisfiability by deriving the empty clause, and can check validity by negating the sentence(s) we want to check for validity and deriving the empty clause.

Equivalence Theorem

Theorem: A sentence ϕ and a sentence ψ are *logically equivalent* if and only if the sentence $(\phi \Leftrightarrow \psi)$ is *valid*.

$\neg(p \wedge q)$ is logically equivalent to $(\neg p \vee \neg q)$
if and only if
$(\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q))$

Upshot: We can determine equivalence of sentences by checking validity of a single sentence.

Upshot: We can demonstrate validity of a biconditional by checking equivalence of the constituents.

Unsatisfiability Theorem

Theorem: $\Delta \models \varphi$ if and only if $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Proof: Suppose that $\Delta \models \varphi$. If an interpretation satisfies Δ, then it must also satisfy φ. But then it cannot satisfy $\neg\varphi$. Therefore, $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Suppose that $\Delta \cup \{\neg\varphi\}$ is unsatisfiable. Then every interpretation that satisfies Δ must *fail* to satisfy $\neg\varphi$, i.e. it must satisfy φ. Therefore, $\Delta \models \varphi$.

Upshot: We can determine logical entailment between sentences by checking unsatisfiability of a set of sentences.

Translation: Assume false and show contradiction.

Deduction Theorem

Theorem: A sentence ϕ *logically entails* a sentence ψ if and only if $(\phi \Rightarrow \psi)$ is *valid*.

More generally, a finite set of sentences $\{\phi_1, \ldots, \phi\}$ logically entails ϕ if and only if the compound sentence $(\phi_1 \wedge \ldots \wedge \phi_n \Rightarrow \phi)$ is valid.

$\{(p \Rightarrow q), (m \Rightarrow p \vee q)\} \models (m \Rightarrow q)$?

Is $((p \Rightarrow q) \wedge (m \Rightarrow p \vee q) \Rightarrow (m \Rightarrow q))$ valid?

Upshot: We can determine logical entailment between sentences by checking validity of a single sentence. *And vice versa*.

Consistency Theorem

Theorem: A sentence ϕ is logically *consistent* with a sentence ψ if and only if the sentence $(\phi \wedge \psi)$ is *satisfiable*. More generally, a sentence ϕ is logically consistent with a finite set of sentences $\{\phi_1, \ldots, \phi_n\}$ if and only if the compound sentence $(\phi_1 \wedge \ldots \wedge \phi_n \wedge \phi)$ is satisfiable.

Is $(p \vee q)$ consistent with $(\neg p \vee \neg q)$?
Is $((p \vee q) \wedge (\neg p \vee \neg q))$ satisfiable?

Upshot: We can determine consistency of sentences by checking satisfiability of a single sentence.

Propositional Analysis (Practice Questions)

Friday, October 14, 2022 4:07 PM

[Question 3](#)
[Question 3](#)

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$$\Delta = \{p\}$$

$$\Gamma = \{p, \neg p\}$$

$$\Delta \subseteq \Gamma$$

$$\Delta \cap \Gamma = \{p\}$$

Schemas/Schemata

Monday, October 17, 2022 8:06 PM

Schema: an expression satisfying the grammatical rules of our language except for the occurrence of *metavariables* (Greek letters) in place of subexpressions

Schema instance: a sentence obtained by consistently substituting *sentences* for the metavariables in the schema.

Valid schema: a schema denoting an infinite set of sentences, all of which are valid. (I.e. a consistent replacement of the metavariables with sentences will always yield a valid sentence.)

Rules of Inference

Monday, October 17, 2022 8:06 PM

Rule of inference: 0 or more schemas called *premises* and one or more additional schemas called *conclusions*.

Direct Proof

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Direct proof of a conclusion from a set of premises: a sequence of sentences terminating in the conclusion in which each item is

1. A premise,
2. An instance of a valid schema, or
3. The result of applying a rule of inference to earlier items in the sequence

Application of a rule of inference: if the premises of a rule can be matched with prior lines of a proof, then the conclusions can be derived.

(Note that metavariables in the rule must match *entire sentences*, not just parts of sentences.)

* Axiom Schemata *

Monday, October 17, 2022 8:48 PM

Axiom Schemata: think of them as rules of inference with no premises.
We will choose our axiom schemata such that they are all valid.

Rules and Axiom Schemata

Axiom Schemata as 0-ary Rules of Inference

$$\varphi \Rightarrow \varphi \qquad\qquad \frac{}{\varphi \Rightarrow \varphi}$$

Rules of Inference as Axiom Schemata

$$\frac{\neg\psi \Rightarrow \neg\varphi}{\varphi \Rightarrow \psi} \qquad\qquad (\varphi \Rightarrow \psi) \Rightarrow (\neg\psi \Rightarrow \neg\varphi)$$

NB: We must keep at least one rule of inference. By convention, we retain Implication Elimination.

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*** Hilbert System ***

Monday, October 17, 2022 8:44 PM

<u>Rule of inference</u>

Implication Elimination (IE)

$\varphi \Rightarrow \psi$

$\underline{\varphi \qquad}$

ψ

<u>Axiom Schemata</u>

Implication Creation (IE)

$\varphi \Rightarrow (\psi \Rightarrow \varphi)$

Implication Distribution (ID)

$(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$

Implication Reversal (IR)

$(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)$

$(\psi \vee \neg\varphi)$

$(\neg\varphi \vee \psi)$

Provability

Monday, October 17, 2022 8:57 PM

Provability

If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R, we say that ϕ is *provable* from Δ using R.

We usually write this as $\Delta \vdash_R \phi$, using the provability operator $\vdash$ (which is sometimes called *single turnstile*). (If the set of rules is clear from context, we usually drop the subscript, writing just $\Delta \vdash \phi$.)

$$\{p \Rightarrow q, q \Rightarrow r\} \vdash (p \Rightarrow r)$$

Soundness and Completeness

Monday, October 17, 2022 8:59 PM

Soundness and Completeness

A proof system is *sound* if and only if every provable conclusion is logically entailed.

$$\text{If } \Delta \vdash \phi, \text{ then } \Delta \vDash \phi.$$

A proof system is *complete* if and only if every logically entailed conclusion is provable.

$$\text{If } \Delta \vDash \phi, \text{ then } \Delta \vdash \phi.$$

Note that soundness and completeness are properties possessed by proof systems.

Taken together, these properties mean that a proof system and the truth table method succeed in exactly the same cases.

The Hilbert, Fitch, and Resolution proof systems are all sound and complete.

Direct Proofs (Practice Question)

Sunday, October 16, 2022 11:54 AM

Premises:

r

~(p => q) => ~(q => r)

Goal:

p => q

Hilbert (stanford.edu)

Note

Monday, October 17, 2022 10:14 PM

We will be discussing natural deduction specifically in the context of the Fitch proof system.

Deduction Theorem

Monday, October 17, 2022 10:10 PM

Deduction Theorem: $\Delta \models (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \models \psi$.

Corollary: $\Delta \vdash (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \vdash \psi$.

Corollary interpretation: the same holds for provability in sound and complete proof systems.

Key idea: To prove an implication, you can make the antecedent an additional premise/assumption and then derive the consequent.

Structured Proof

Monday, October 17, 2022 10:18 PM

A **structured proof** of a conclusion from a set of premises is a sequence of (possibly nested) sentences terminating in the *conclusion at the top level* of the proof in which each item is

1. A premise (at the top level),
2. An assumption, or
3. The result of applying an *ordinary rule of inference* or a *structured rule of inference* to earlier items in the sequence.

Constraint: ordinary rules of inference can only be applied in a subproof to items that occur earlier in that subproof or in a superproof of that subproof.
Similarly, an ordinary rule of inference can only be applied at the top level of a proof to other items that are in the top level.

Structured Rule of Inference

Monday, October 17, 2022 10:23 PM

Structured rule of inference: a rule of inference where one of the premises is of the form $\varphi \vdash \psi$.

Example:

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Translation: If there is a subproof with assumption ϕ and conclusion ψ, then conclude $\phi \Rightarrow \psi$ outside the subproof.

* Making use of the Deduction Theorem with Subproofs *

Monday, October 17, 2022 10:15 PM

The Fitch proof system takes advantage of the Deduction Theorem via *subproofs*.

$$\text{Deduction Theorem: } \Delta \vDash (\varphi \Rightarrow \psi) \text{ if and only if } \Delta \cup \{\varphi\} \vDash \psi.$$
$$\text{Corollary: } \Delta \vdash (\varphi \Rightarrow \psi) \text{ if and only if } \Delta \cup \{\varphi\} \vdash \psi.$$

Subproofs in Fitch allow us to derive implications.

A subproof begins if-and-only-if we make an assumption. We can begin a subproof at any point in a Fitch proof.

We can assume anything, but this doesn't make the assumption true at the top level of the proof. Instead, an assumption allows us to act as though the assumption is true *within the subproof* that begins with our assumption, to see what that assumption would allow us to derive.

When we exit a subproof, we do so using our single structured rule of inference: **Implication Introduction**. This results in leaving the subproof and introducing an implication after and outside of the subproof.
The *antecedent of the new implication is the first line of the subproof*, and the *consequent is the final line of the subproof.*

Key idea/tip: Whenever you make an assumption, it should be with the intent of deriving an implication upon exiting the subproof.

Note that, since a structured proof must end in our conclusion at the top level of the proof, **we must eventually exit the subproof**. That is, a Fitch proof will never end within a subproof.

Implication Introduction

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Translation: If there is a subproof with assumption ϕ and conclusion ψ, then conclude $\phi \Rightarrow \psi$ outside the subproof.

Axiom Schemata No More

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Note that since Fitch is sound and complete, we can use subproofs and Implication Introduction to derive any implication that is entailed by our premises.

And since *valid implications* are entailed by all sets of premises, we can always derive valid implications using Fitch.

But since the axiom schemata in Hilbert are specifically used to introduce valid implications, there is no need for them in Fitch!

Fitch Ordinary Rules of Inference

Monday, October 17, 2022 10:52 PM

[Slides 18-22]

Proof Tips

Monday, October 17, 2022 10:54 PM

- If you need to assume multiple things, you can functionally do this by assuming a conjunction.
    - Just know that once you exit the subproof via Implication Introduction, the new implication will have that conjunction for its antecedent.
- From before: Whenever you make an assumption, it should be with the intent of deriving an implication upon exiting the subproof.

(Not an algorithm, but frequently helpful to try these.)
From class: Slides 39-42

Follow-up to Tip 3:
We don't have a lot of ways to derive disjunctions in Fitch. So, it's often helpful to apply Tip 5 and try to prove a disjunction by contradiction.

Natural Deduction (Practice Question)

Sunday, October 16, 2022 12:23 PM

Premise:

p <=> q

Goal:

~p <=> ~q

Premise:

~p | ~q

Goal:

~(p & q)

[Fitch (stanford.edu)]

Refutation Proof

Monday, October 17, 2022 11:08 PM

A refutation proof is a sequence of sentences that terminates in some form of contradiction, in which each sentence is

- a premise,
- the negation of a desired conclusion,
- the result of applying a rule of inference to earlier items in the sequence.

Therefore, a refutation proof is a proof by contradiction.

* Clausal Form *

Monday, October 17, 2022 11:27 PM

Resolution only works on expressions that are in clausal form.

Clausal Form

A *literal* is either an atomic sentence or a negation of an atomic sentence.

$$p, \neg p$$

A *clausal sentence* is either a literal or a disjunction of literals.

$$p, \quad \neg p, \quad p \vee \neg q$$

A *clause* is a set of literals.

$$\{p\}, \{\neg p\}, \{p, \neg q\}$$

<u>Semantics</u>

A clause is functionally a disjunction of literals.
If at least one of its elements is true, then it is true.

By these semantics, the empty clause is unsatisfiable.
Why?
Because a disjunction is only true if at least one of its elements is true, but there are no elements.

Resolution Principle

Monday, October 17, 2022 11:16 PM

The Propositional Resolution proof system that we're using for creating refutation proofs uses a single rule of inference: the resolution principle.

Resolution Principle

$$\frac{\begin{array}{l}\{\phi_1, \ldots, \cancel{\chi}, \ldots, \phi_m\} \\ \{\psi_1, \ldots, \neg\cancel{\chi}, \ldots, \psi_n\}\end{array}}{\{\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n\}}$$

$$\frac{\begin{array}{l} p \vee \neg \cancel{q} \\ r \vee \cancel{q} \end{array}}{p \vee r}$$

$$(\neg p \vee q)$$
$$(p \vee \neg q)$$

$$(q \vee \neg q)$$
$$(\neg p \vee p)$$

Resolution Derivation

Monday, October 17, 2022 11:33 PM

A **resolution derivation** of a conclusion from a set of premises: a finite sequence of clauses terminating in the conclusion in which each clause is either a premise or the result of applying the resolution principle to earlier elements of the sequence.

* Resolution Determines Satisfiability *

Monday, October 17, 2022 11:35 PM

Resolution determines satisfiability of premises.
That is, if a set of clauses is unsatisfiable, then it is possible to derive the empty clause using the resolution principle.

Our strategy for generating refutation proofs is motivated by the unsatisfiability theorem.
Unsatisfiability theorem: $\Delta \vDash \varphi$ if and only if $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

That is, the resolution method generates a refutation proof of a conclusion φ by adding the $\neg\varphi$ to the premises, then deriving the empty clause.

Refutation Proofs (Practice Questions)

Sunday, October 16, 2022 12:58 PM

<u>Premise:</u>

$(p \Rightarrow q) \Rightarrow (p \land \neg q)$,

<u>Goal:</u>

$\neg(p \Rightarrow q)$

<u>Premises:</u>

$p \Rightarrow (q \lor r)$

$r \Rightarrow q$

<u>Goal:</u>

$\neg p \lor q$

[Robinson (stanford.edu)]

Conversion to Clausal Form (INDO)

Monday, October 17, 2022 11:15 PM

Slides 11-16

Conversion to Clausal form:

Implications out
Negations in
Distribution
Operators out

Elimination Strategies

Monday, October 17, 2022 8:30 PM

[Slides 58-65]

Most important:
Identical Clause Elimination
Tautology Elimination

Additional:
Pure Literal Elimination
Subsumption Elimination