Introduction to Logic

Open Worlds

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Objects with Multiple Names

People (Nicknames):

\[
\begin{align*}
michael & \iff \bullet \\
mike & \iff \text{stick figure}
\end{align*}
\]

Arithmetic:

\[
\begin{align*}
\text{plus}(s(0), s(0)) & \iff \\
\text{times}(s(s(0)), s(0)) & \iff 2 \\
s(s(0)) & \iff
\end{align*}
\]
Objects with No Names

People

michael ↔

maureen ↔

Unnamed People
Named real numbers (countably many):

- 123
- 34.12
- $\pi$
- $e$

All the others (uncountably many):

- 3.141544878723489184093893477809489084...
- 6.878989783975975738975379875837593358...

How many floating point numbers are there?
First Order Logic

Syntax
   Identical to Herbrand Logic

Semantics
   Herbrand Logic - defined in terms of language
   First Order Logic - defined in terms of external worlds
Language of First Order Logic

\[ p(a) \quad q(b, c) \]

\[ \neg p(b) \quad \forall x. (p(x) \implies q(x, f(x))) \]

\[ p(c) \lor p(d) \quad \exists x. q(x, d) \]

Identical to Syntax of Functional Logic
Interpretations

A vocabulary is a set of symbols.

\[ \{a, b, f, q\} \]

A universe of discourse is an arbitrary set of objects.

\[ \{ \text{objects} \} \]

An interpretation is an assignment to symbols in language.

\[ a=\text{object}, \quad b=\text{object} \]

\[ f=\{ \text{function} \} \]

\[ q=\{ \text{pair} \} \]
A vocabulary is a set of symbols.

\{a, b, f, q\}

A *universe of discourse* is an *arbitrary* set of objects.

\{1, 2, 3, 4\}

An *interpretation* is an assignment to symbols in language.

\[a=1, \ b=2\]

\[f=\{1\rightarrow 2, \ 2\rightarrow 3, \ldots\}\]

\[q=\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \ldots\}\]

Entailment defined in terms of *all interpretations over all possible universes of all possible sizes*
A *variable assignment* for a universe of discourse $U$ is a function assigning variables to objects in $U$.

$v: \text{Variable} \rightarrow U$

Universe of Discourse:
$U = \{1, 2, 3\}$

Example:

Example:

\[
\begin{align*}
  v(x) &= 1 \\
  v(y) &= 2 \\
  v(z) &= 3 \\
  v(x) &= 2 \\
  v(y) &= 2 \\
  v(z) &= 2 
\end{align*}
\]
A value assignment $s_{iv}$ based on interpretation $i$ and variable assignment $v$ is a mapping from the terms of the language into the universe of discourse.

\[
\begin{align*}
    s_{iv}(\sigma) &= i(\sigma) \\
    s_{iv}(\upsilon) &= v(\upsilon) \\
    s_{iv}(\pi(\tau_1, \ldots, \tau_n)) &= i(\pi)(s_{iv}(\tau_1), \ldots, s_{iv}(\tau_n))
\end{align*}
\]
A truth assignment satisfies a relational sentence if and only if the tuple of objects denoted by the arguments is a member of the relation denoted by the relation constant.

\[ t_{iv}(\rho(\tau_1, \ldots, \tau_n)) = true \text{ if } \langle s_{iv}(\tau_1), \ldots, s_{iv}(\tau_1) \rangle \in i(\rho) \]
\[ = false \text{ otherwise} \]
$t_{iv}(-\varphi) = true \iff t_{iv}(\varphi) = false$

$t_{iv}(\varphi \land \psi) = true \iff t_{iv}(\varphi) = true \land t_{iv}(\psi) = true$

$t_{iv}(\varphi \lor \psi) = true \iff t_{iv}(\varphi) = true \lor t_{iv}(\psi) = true$

$t_{iv}(\varphi \Rightarrow \psi) = true \iff t_{iv}(\varphi) = false \lor t_{iv}(\psi) = true$

$t_{iv}(\varphi \Leftrightarrow \psi) = true \iff t_{iv}(\varphi) = t_{iv}(\psi)$

Logical Sentences
Intuitively, a universally quantified sentence is true if and only if it is true no matter what value we assign to the universally quantified variable.

Intuitively, an existentially quantified sentence is true if and only if it is true for some value of the existentially quantified variable.

Stating these definitions precisely is a little tricky due to the possibility of nested quantifiers.

\[ \forall x. (\exists y. r(x,y) \Rightarrow \forall x. r(x,x)) \]
A version $v[\omega \leftarrow x]$ of a variable assignment $v$ is the variable assignment that agrees with $v$ on all variables except $\omega$, which is assigned the value $x$.

$$v[\omega \leftarrow x](\theta) = x \quad \text{if } \theta = \omega$$
$$v[\omega \leftarrow x](\theta) = v(\theta) \quad \text{if } \theta \neq \omega$$
Quantified Sentences

A universally quantified sentence is true in interpretation \( i \) and variable assignment \( \nu \) if and only if the scope is true for \( i \) and every version of \( \nu \).

\[
t_{iv}(\forall \omega. \varphi) = true \text{ iff } t_{iv[\omega \leftarrow x]}(\varphi) = true \text{ for all } x \in |i|.
\]

An existentially quantified sentence is true in interpretation \( i \) and variable assignment \( \nu \) if and only if the scope is true for \( i \) and some version of \( \nu \).

\[
t_{iv}(\exists \omega. \varphi) = true \text{ iff } t_{iv[\omega \leftarrow x]}(\varphi) = true \text{ for some } x \in |i|.
\]
First Order Logic: A universally quantified sentence is true in interpretation $i$ and variable assignment $v$ if and only if the scope is true for $i$ and every version of $v$ compatible with interpretation $i$.

Herbrand Logic: A universally quantified sentence is true in a truth assignment if and only if every instance is true.
In Herbrand Logic, if $\Delta \models p(\tau)$ for every ground term $\tau$, does $\Delta \models \forall x.p(x)$?

Yes.
In Herbrand Logic, if $\Delta \models p(\tau)$ for every ground term $\tau$, does $\Delta \models \forall x.p(x)$?

Yes.

In First-Order Logic, if $\Delta \models p(\tau)$ for every ground term $\tau$, does $\Delta \models \forall x.p(x)$?

No. There can be objects without names.
Can we describe uncountable world in Herbrand Logic?

No. There only countably many terms and countably many ground sentences in our language.

Upshot: It is not possible to axiomatize uncountable worlds in Herbrand Logic.
Can we describe uncountable worlds in First-Order Logic?

Yes. There can be objects without names.

Lowenhein-Skolem Theorem: If a set of sentences in First Order Logic has a model of one infinite cardinality, then it has a model of every infinite cardinality. (This striking result is true but not obvious.)

*Upshot: It is not possible to write a sentence in First Order Logic that is true in an uncountable world and not true in any countably infinite world or vice-versa.*
Peano Arithmetic

Transitive Closure

Both of these are finite axiomatizations and are complete (i.e. they precisely define which sentences are true and which are false). There are no non-standard models.
First-Order Logic (FOL) theories with infinite universes have nonstandard models (unintended models that cannot be excluded).

Upshot: FOL is weaker than Herbrand Logic. Some notions that can be defined exactly in Herbrand Logic cannot be defined in FOL without allowing nonstandard models, e.g. Peano Arithmetic, transitive closure.
There is a proof procedure for First Order Logic that is both sound and complete. (Spoiler Alert: Fitch without Domain Closure and Induction does the trick.)

Moreover, by systematically applying the procedure, possible to compute all logical consequences of any enumerable set of premises. (Apply Fitch to produce all finite proofs in systematic way.)

Upshot: provability and logical entailment are semi-decidable (though not decidable).
The axiomatization of Peano Arithmetic in Herbrand Logic completely defines Peano Arithmetic.

If Herbrand entailment were semi-decidable, the set of all true sentences would be enumerable.

Godel’s incompleteness theorem tells us that the set of all true sentences of Peano arithmetic is not computably enumerable.

Consequently, there is no complete (semi-decidable) proof procedure for Herbrand Logic.
Theorem: Any sound proof procedure for First Order Logic is sound for Herbrand Logic.

Even though there is no complete proof procedure, Herbrand logic is not weaker. In fact, Herbrand logic is stronger than FOL. There are simply more things that are true.

We cannot prove them all, but we can prove everything we could prove in First Order Logic; and, by building in induction, we can prove more things.
Summary

First Order Logic:
- Compact
- Complete proof procedure
- Semi-decidable

Herbrand Logic:
- Not compact
- No complete proof procedure
- Not even semi-decidable

Comparison of Herbrand Logic over First Order Logic:
Herbrand Semantics simpler and more intuitive
FOL can be used to describe uncountable worlds
More things definable in Herbrand Logic
Greater inferential power in HL but not complete