# Introduction to Logic
## *Induction*

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# Truth Table for Relational Logic

$$\{p(a) \vee p(b), \forall x.(p(x) \Rightarrow q(x))\} \models \exists x.q(x)?$$

| $p(a)$ | $p(b)$ | $q(a)$ | $q(b)$ | $p(a) \vee p(b)$ | $\forall x.(p(x) \Rightarrow q(x))$ | $\exists x.q(x)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 |

# Relational Logic Sizes

Object constants: $n$
Binary relation constants: $k$
Factoids in Herbrand Base: $k*n^2$
Truth Assignments: $2^{k*n^2}$

Object constants: 4
Binary relation constants: 4
Factoids in Herbrand Base: 64
Truth Assignments: $2^{64}$ = 18,446,744,073,709,551,616

*Upshot: Truth tables impractical.*
*Proofs more practical.*

# Functional Logic Sizes

Object constants: $n \geq 1$

Functions constants: $j \geq 1$

Relation constants: $k \geq 1$

Factoids in Herbrand Base: *countably infinite*

Truth Assignments: *uncountably infinite*

*Upshot: Truth tables impossible.*

*Proofs are our only hope.*

# Fitch System for Relational Logic

Negation Introduction
Negation Elimination

And Introduction
And Elimination

Or Introduction
Or Elimination

Assumption
Implication Introduction
Implication Elimination

Biconditional Introduction
Biconditional Elimination

Universal Introduction
Universal Elimination

Existential Introduction
Existential Elimination

Domain Closure

# Fitch System for Functional Logic

Negation Introduction
Negation Elimination

And Introduction
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Or Introduction
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Assumption
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Biconditional Elimination

Universal Introduction
Universal Elimination

Existential Introduction
Existential Elimination

Domain Closure
Induction *New!!*

# Overview

# Induction

Induction is reasoning from the specific to the general.

If various instances of a schema are true and there are no counterexamples, we are tempted to conclude a universally quantified version of the schema.

$$\begin{array}{c} p(a) \Rightarrow q(a) \\ p(b) \Rightarrow q(b) \\ p(c) \Rightarrow q(c) \end{array} \quad \rightarrow \quad \forall x.(p(x) \Rightarrow q(x))$$

# Lucky Guess

Definition

$$f(1) = 1$$
$$f(x + 1) = f(x) + 2*x + 1$$

Data

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | $1$ | = | 1 | = | $1^2$ |
| 2 | $1+3$ | = | 4 | = | $2^2$ |
| 3 | $1+3+5$ | = | 9 | = | $3^2$ |
| 4 | $1+3+5+7$ | = | 16 | = | $4^2$ |
| 5 | $1+3+5+7+9$ | = | 25 | = | $5^2$ |

Conjecture

$$f(x) = x^2$$

*In this case, the answer is correct. Lucky Guess.*

# Not So Lucky Guess

Data:

$$2^{2^1} + 1 = 2^4 + 1 = 5$$
$$2^{2^2} + 1 = 2^4 + 1 = 17$$
$$2^{2^3} + 1 = 2^8 + 1 = 257$$
$$2^{2^4} + 1 = 2^{16} + 1 = 65537$$

"Theorem" by Fermat (1601-1665):

$$prime(2^{2^x} + 1)$$

Fact discovered (mercifully) after his death:

$$2^{2^5} + 1 = 4{,}294{,}967{,}297 = 641 * 6{,}700{,}417$$

*Oops*.

# Domain Closure

$$\frac{\begin{array}{c}\phi[\sigma_1] \\ \ldots \\ \phi[\sigma_n]\end{array}}{\forall \upsilon . \phi[\upsilon]} \quad \left.\right\} \textit{every ground term}$$

# Finite Example

$$\frac{\begin{array}{c} p(a) \Rightarrow q(a) \\ p(b) \Rightarrow q(b) \\ p(c) \Rightarrow q(c) \\ p(d) \Rightarrow q(d) \end{array}}{\forall x.(p(x) \Rightarrow q(x))}$$

# Infinite Case

What happens when we have infinitely many terms?

$$\frac{\begin{array}{c} p(a) \Rightarrow q(a) \\ p(s(a)) \Rightarrow q(s(a)) \\ p(s(s(a))) \Rightarrow q(s(s(a))) \\ p(s(s(s(a)))) \Rightarrow q(s(s(s(a)))) \\ \dots \end{array}}{\forall x.(p(x) \Rightarrow q(x))}$$

# Types of Induction

Linear Induction
    induction on sequences

Tree Induction
    induction on trees

Structural Induction
    induction on complex structures

# Linear Induction

# Linear Worlds

$$a \rightarrow s(a) \rightarrow s(s(a)) \rightarrow s(s(s(a))) \rightarrow \ldots$$

# Linear Worlds

Linear World:

$$a \rightarrow s(a) \rightarrow s(s(a)) \rightarrow s(s(s(a))) \rightarrow \ldots$$

The object constant is called the *base element*, and the function constant is called the *successor function*.

# Induction with Linear Worlds

Linear World:

$$a \rightarrow s(a) \rightarrow s(s(a)) \rightarrow s(s(s(a))) \rightarrow \ldots$$

Linear Induction:

(1) If we can show our *base element* has some property

*and*

(2) if we can show that, whenever an arbitrary element has that property, its *successor* has that property,

then we can conclude that *every* element has that property.

# Dominoes

# Dominoes Example

Object constant: *a*
Unary function constant: *s*
Unary relation constant: *falls*

# Dominoes Example

Object constant: $a$
Unary function constant: $s$
Unary relation constant: $falls$

Axioms:

$$falls(a)$$
$$\forall x.(falls(x) \Rightarrow falls(s(x)))$$

# Dominoes Example

Object constant: *a*
Unary function constant: *s*
Unary relation constant: *falls*

Axioms:

$$falls(a)$$
$$\forall x.(falls(x) \Rightarrow falls(s(x)))$$

Conclusion:

$$\forall x.falls(x)$$

GUINNESS WORLD RECORDS™

# OFFICIALLY

# Linear Induction

$$\frac{\phi[a] \qquad \forall x.(\phi[x] \Rightarrow \phi[s(x)])}{\forall x.\phi[x]}$$

# Linear Induction

Base Case

$$\phi[a]$$

$$\forall x.(\phi[x] \Rightarrow \phi[s(x)])$$

$$\overline{\forall x.\phi[x]}$$

# Linear Induction

Base Case

Inductive Case

$$\frac{\phi[a] \qquad \forall x.(\phi[x] \Rightarrow \phi[s(x)])}{\forall x.\phi[x]}$$

# Linear Induction

Base Case

$$\phi[a]$$

$$\forall x.(\phi[x] \Rightarrow \phi[s(x)])$$

Inductive Case

$$\frac{}{\forall x.\phi[x]}$$

Overall Conclusion

# Linear Induction

Inductive Hypothesis

Base Case

$$\frac{\phi[a] \qquad \forall x.(\phi[x] \Rightarrow \phi[s(x)])}{\forall x.\phi[x]}$$

Inductive Case

Overall Conclusion

# Linear Induction

Inductive Hypothesis

Base Case

Inductive Conclusion

$$\frac{\phi[a] \qquad \forall x.(\phi[x] \Rightarrow \phi[s(x)])}{\forall x.\phi[x]}$$

Inductive Case

Overall Conclusion

# Whole Numbers

Object constant: 0

Unary function constant: $s$

$$0 \rightarrow s(0) \rightarrow s(s(0)) \rightarrow s(s(s(0))) \rightarrow \ldots$$

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Object constant: 0

Unary function constant: $s$

$$0 \rightarrow s(0) \rightarrow s(s(0)) \rightarrow s(s(s(0))) \rightarrow \ldots$$

Unary relation constants: $even$, $odd$

# Whole Numbers

Object constant: 0

Unary function constant: $s$

$$0 \rightarrow s(0) \rightarrow s(s(0)) \rightarrow s(s(s(0))) \rightarrow \ldots$$

Unary relation constants: $even$, $odd$

Axioms:

$$even(0)$$

$$\forall x.(even(x) \Rightarrow odd(s(x))$$

$$\forall x.(odd(x) \Rightarrow even(s(x))$$

# Whole Numbers

Object constant: 0

Unary function constant: $s$

$$0 \rightarrow s(0) \rightarrow s(s(0)) \rightarrow s(s(s(0))) \rightarrow \ldots$$

Unary relation constants: $even$, $odd$

Axioms:

$$even(0)$$

$$\forall x.(even(x) \Rightarrow odd(s(x))$$

$$\forall x.(odd(x) \Rightarrow even(s(x))$$

Goal:

$$\forall x.(even(x) \vee odd(x))$$

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |

# Proof

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|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $\mid even(c) \vee odd(c)$ | Assumption |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $\mid even(c) \vee odd(c)$ | Assumption |
| 6. | $\mid \mid even(c)$ | Assumption |

# Proof

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|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $\quad even(c) \vee odd(c)$ | Assumption |
| 6. | $\quad\quad even(c)$ | Assumption |
| 7. | $\quad\quad even(c) \Rightarrow odd(s(c))$ | UE: 2 |
| 8. | $\quad\quad odd(s(c))$ | IE: 7, 6 |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $\quad even(c) \vee odd(c)$ | Assumption |
| 6. | $\quad\quad even(c)$ | Assumption |
| 7. | $\quad\quad even(c) \Rightarrow odd(s(c))$ | UE: 2 |
| 8. | $\quad\quad odd(s(c)$ | IE: 7, 6 |
| 9. | $\quad\quad even(s(c)) \vee odd(s(c))$ | OI: 8 |
| 10. | $\quad even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6, 9 |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $\quad even(c) \vee odd(c)$ | Assumption |
| | $\quad\quad ...$ | |
| 10. | $\quad even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6, 9 |

# Proof

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| 1 | $even(0)$ | Premise |
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| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $\mid even(c) \vee odd(c)$ | Assumption |
| | $\mid$ ... | |
| 10. | $\mid even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6, 9 |
| 11. | $\mid \mid odd(c)$ | Assumption |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $even(c) \vee odd(c)$ | Assumption |
| | ... | |
| 10. | $even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6, 9 |
| 11. | $odd(c)$ | Assumption |
| 12. | $odd(c) \Rightarrow even(s(c))$ | UI: 3 |
| 13. | $even(s(c))$ | IE: 12, 11 |
| 14. | $even(s(c)) \vee odd(s(c))$ | OI: 13 |
| 15. | $odd(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 11, 15 |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $even(c) \vee odd(c)$ | BE: 4 |
| | ... | |
| 10. | $even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6,9 |
| | ... | |
| 15. | $odd(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 11,14 |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $even(c) \vee odd(c)$ | BE: 4 |
| | ... | |
| 10. | $even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6,9 |
| | ... | |
| 15. | $odd(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 11,14 |
| 16. | $even(s(c)) \vee odd(s(c))$ | OE: 5, 10, 15 |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $\quad\vert\; even(c) \vee odd(c)$ | BE: 4 |
| | $\quad\vert\; \quad ...$ | |
| 10. | $\quad\vert\; even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6,9 |
| | $\quad\vert\; \quad ...$ | |
| 15. | $\quad\vert\; odd(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 11,14 |
| 16. | $\quad\vert\; even(s(c)) \vee odd(s(c))$ | OE: 5, 10, 15 |
| 17. | $even(c) \vee odd(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 5, 16 |

# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \lor odd(0)$ | OI: 1 |
| 5. | $\vert$ $even(c) \lor odd(c)$ | BE: 4 |
| | $\vert$ ... | |
| 10. | $\vert$ $even(c) \Rightarrow even(s(c)) \lor odd(s(c))$ | II: 6,9 |
| | $\vert$ ... | |
| 15. | $\vert$ $odd(c) \Rightarrow even(s(c)) \lor odd(s(c))$ | II: 11,14 |
| 16. | $\vert$ $even(s(c)) \lor odd(s(c))$ | OE: 5, 10, 15 |
| 17. | $even(c) \lor odd(c) \Rightarrow even(s(c)) \lor odd(s(c))$ | II: 5, 16 |
| 18. | $\forall x.(even(x) \lor odd(x) \Rightarrow even(s(x)) \lor odd(s(x)))$ | UI: 17 |

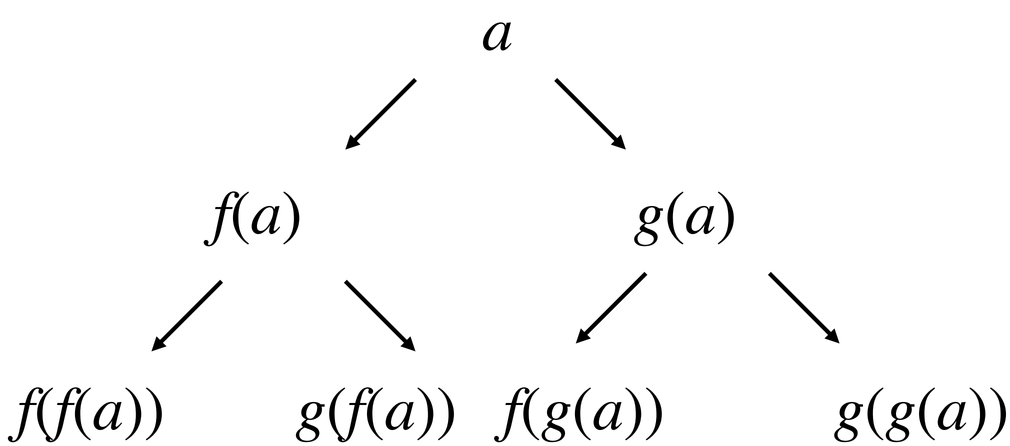
# Proof

| | | |
|---|---|---|
| 1 | $even(0)$ | Premise |
| 2. | $\forall x.(even(x) \Rightarrow odd(s(x)))$ | Premise |
| 3. | $\forall x.(odd(x) \Rightarrow even(s(x)))$ | Premise |
| 4. | $even(0) \vee odd(0)$ | OI: 1 |
| 5. | $even(c) \vee odd(c)$ | BE: 4 |
| | ... | |
| 10. | $even(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 6,9 |
| | ... | |
| 15. | $odd(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 11,14 |
| 16. | $even(s(c)) \vee odd(s(c))$ | OE: 5, 10, 15 |
| 17. | $even(c) \vee odd(c) \Rightarrow even(s(c)) \vee odd(s(c))$ | II: 5, 16 |
| 18. | $\forall x.(even(x) \vee odd(x) \Rightarrow even(s(x)) \vee odd(s(x)))$ | UI: 17 |
| 19. | $\forall x.(even(x) \vee odd(x))$ | Ind: 4, 18 |

# Tree Induction

# Tree Languages

Tree-Like World:




Languages like this are called *tree languages*.

# Tree Induction

$$\frac{\begin{array}{l}\phi[a] \\ \forall \mu.(\phi[\mu] \Rightarrow \phi[f(\mu)]) \\ \forall \mu.(\phi[\mu] \Rightarrow \phi[g(\mu)])\end{array}}{\forall \upsilon.\phi[\upsilon]}$$

# Canine Ancestry Example

Object constant: *rex*
Unary function constants: *f*, *g*
Unary relation constant: *p*

# Canine Ancestry Example

Object constant: *rex*
Unary function constants: *f*, *g*
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Axioms:

$$p(rex)$$
$$\forall x.(p(x) \Leftrightarrow p(f(x)) \wedge p(g(x)))$$

# Canine Ancestry Example

Object constant: *rex*
Unary function constants: *f*, *g*
Unary relation constant: *p*

Axioms:

$$p(rex)$$
$$\forall x.(p(x) \Leftrightarrow p(f(x)) \wedge p(g(x)))$$

Goal:

$$\forall x.p(x)$$

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
| 2. | $\forall x.(p(x) \Leftrightarrow p(f(x)) \wedge p(g(x)))$ | Premise |

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
| 2. | $\forall x.(p(x) \Leftrightarrow p(f(x)) \wedge p(g(x)))$ | Premise |
| 3. | $p(c) \Leftrightarrow p(f(c)) \wedge p(g(c))$ | UE: 2 |
| 4. | $p(c) \Rightarrow p(f(c)) \wedge p(g(c))$ | BE: 3 |

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
| 2. | $\forall x.(p(x) \Leftrightarrow p(f(x)) \wedge p(g(x)))$ | Premise |
| 3. | $p(c) \Leftrightarrow p(f(c)) \wedge p(g(c))$ | UE: 2 |
| 4. | $p(c) \Rightarrow p(f(c)) \wedge p(g(c))$ | BE: 3 |
| 5. | $\quad \mid p(c)$ | Assumption |

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
| 2. | $\forall x.(p(x) \Leftrightarrow p(f(x)) \wedge p(g(x)))$ | Premise |
| 3. | $p(c) \Leftrightarrow p(f(c)) \wedge p(g(c))$ | UE: 2 |
| 4. | $p(c) \Rightarrow p(f(c)) \wedge p(g(c))$ | BE: 3 |
| 5. | $\quad \mid p(c)$ | Assumption |
| 6. | $\quad \mid p(f(c)) \wedge p(g(c))$ | IE: 4, 5 |

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
| 2. | $\forall x.(p(x) \Leftrightarrow p(f(x)) \wedge p(g(x)))$ | Premise |
| 3. | $p(c) \Leftrightarrow p(f(c)) \wedge p(g(c))$ | UE: 2 |
| 4. | $p(c) \Rightarrow p(f(c)) \wedge p(g(c))$ | BE: 3 |
| 5. | $\quad\vert\ p(c)$ | Assumption |
| 6. | $\quad\vert\ p(f(c)) \wedge p(g(c))$ | IE: 4, 5 |
| 7. | $\quad\vert\ p(f(c))$ | AE: 6 |

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
| 2. | $\forall x.(p(x) \Leftrightarrow p(f(x)) \land p(g(x)))$ | Premise |
| 3. | $p(c) \Leftrightarrow p(f(c)) \land p(g(c))$ | UE: 2 |
| 4. | $p(c) \Rightarrow p(f(c)) \land p(g(c))$ | BE: 3 |
| 5. | $\quad\vert\ p(c)$ | Assumption |
| 6. | $\quad\vert\ p(f(c)) \land p(g(c))$ | IE: 4, 5 |
| 7. | $\quad\vert\ p(f(c))$ | AE: 6 |
| 8. | $p(c) \Rightarrow p(f(c))$ | II: 5, 7 |
| 9. | $\forall x.(p(x) \Rightarrow p(f(x)))$ | UI: 8 |

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
| 2. | $\forall x.(p(x) \Leftrightarrow p(f(x)) \land p(g(x)))$ | Premise |
| 3. | $p(c) \Leftrightarrow p(f(c)) \land p(g(c))$ | UE: 2 |
| 4. | $p(c) \Rightarrow p(f(c)) \land p(g(c))$ | BE: 3 |
| | $\ldots$ | |
| 9. | $\forall x.(p(x) \Rightarrow p(f(x)))$ | UI: 8 |
| | $\ldots$ | |
| 14. | $\forall x.(p(x) \Rightarrow p(g(x)))$ | UI: 13 |

# Canine Ancestry Proof

| | | |
|---|---|---|
| 1. | $p(rex)$ | Premise |
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| 3. | $p(c) \Leftrightarrow p(f(c)) \wedge p(g(c))$ | UE: 2 |
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| | … | |
| 9. | $\forall x.(p(x) \Rightarrow p(f(x)))$ | UI: 8 |
| | … | |
| 14. | $\forall x.(p(x) \Rightarrow p(g(x)))$ | UI: 13 |
| 15. | $\forall x.p(x)$ | Ind: 1, 9, 14 |

# Linear Induction

Linear Language:

$$a \rightarrow s(a) \rightarrow s(s(a)) \rightarrow s(s(s(a))) \rightarrow \ldots$$

Induction:

$$\frac{\phi[a] \qquad \forall \mu.(\phi[\mu] \Rightarrow \phi[s(\mu)])}{\forall \upsilon.\phi[\upsilon]}$$

# Multiple Object Constants

Linear Language:

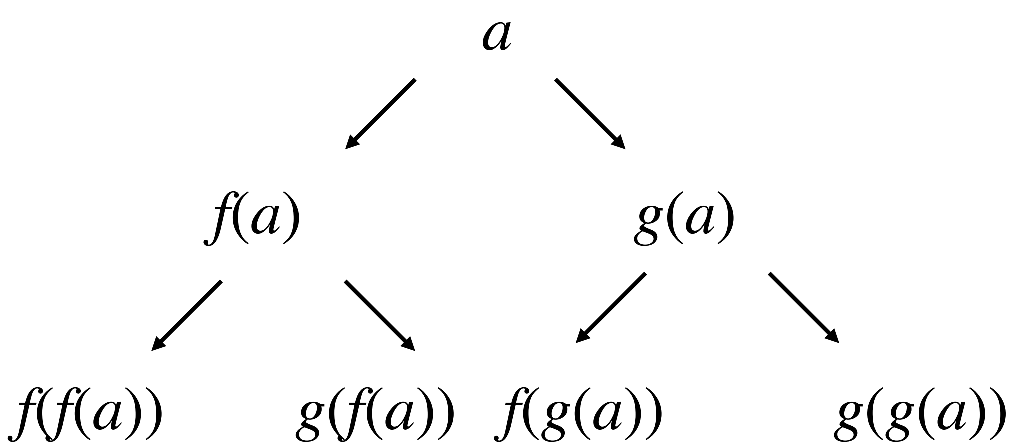
$$a \rightarrow s(a) \rightarrow s(s(a)) \rightarrow s(s(s(a))) \rightarrow \ldots$$
$$b \rightarrow s(b) \rightarrow s(s(b)) \rightarrow s(s(s(b))) \rightarrow \ldots$$

Induction:

$$\frac{\begin{array}{l}\phi[a] \\ \phi[b] \\ \forall \mu.(\phi[\mu] \Rightarrow \phi[s(\mu)])\end{array}}{\forall \upsilon.\phi[\upsilon]}$$

# Tree Induction

Tree Language:

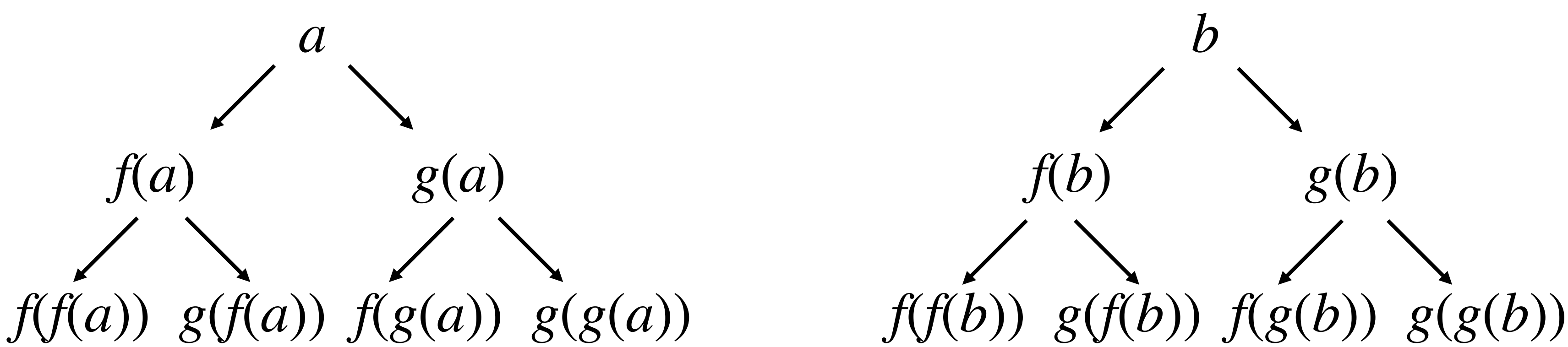



Induction:

$$\frac{\begin{array}{l}\phi[a] \\ \forall \mu.(\phi[\mu] \Rightarrow \phi[f(\mu)]) \\ \forall \mu.(\phi[\mu] \Rightarrow \phi[g(\mu)])\end{array}}{\forall \upsilon.\phi[\upsilon]}$$

# Multiple Object and Function Constants

Tree Language:




Induction:

$$\frac{\begin{array}{l}\phi[a] \\ \phi[b] \\ \forall \mu.(\phi[\mu] \Rightarrow \phi[f(\mu)]) \\ \forall \mu.(\phi[\mu] \Rightarrow \phi[g(\mu)])\end{array}}{\forall \upsilon.\phi[\upsilon]}$$

# Structural Induction

# $n$-ary Function Constant

Object constant: $a$

Binary function constant: $f$

Sample Terms

$a$,

$f(a,a)$,

$f(a,f(a,a))$,

$f(f(a,a),a)$,

$f(f(a,a),f(a,a))$,

$f(a,f(a,f(a,a)))$,

$f(a,f(f(a,a),a))$,

…

# Structural Induction

Object constant: *a*

Binary function constant: *f*

$$\frac{\phi[a] \qquad \forall\lambda.\forall\mu.(\phi[\lambda] \land \phi[\mu] \Rightarrow \phi[f(\lambda,\mu)])}{\forall\upsilon.\phi[\upsilon]}$$

# Structural Induction

Object constants: $a$, $b$
Binary function constant: $f$

$$\frac{\begin{array}{l}\phi[a] \\ \phi[b] \\ \forall\lambda.\forall\mu.(\phi[\lambda] \wedge \phi[\mu] \Rightarrow \phi[f(\lambda,\mu)])\end{array}}{\forall\upsilon.\phi[\upsilon]}$$

# Structural Induction

Object constant: $a$
Binary function constants: $f$, $g$

$$\frac{\begin{array}{l}\phi[a] \\ \forall\lambda.\forall\mu.(\phi[\lambda] \wedge \phi[\mu] \Rightarrow \phi[f(\lambda,\mu)]) \\ \forall\lambda.\forall\mu.(\phi[\lambda] \wedge \phi[\mu] \Rightarrow \phi[g(\lambda,\mu)])\end{array}}{\forall\upsilon.\phi[\upsilon]}$$

# Structural Induction

Object constants: $a$, $b$
Binary function constants: $f$, $g$

$$\frac{\begin{array}{l}\phi[a] \\ \phi[b] \\ \forall\lambda.\forall\mu.(\phi[\lambda] \wedge \phi[\mu] \Rightarrow \phi[f(\lambda,\mu)]) \\ \forall\lambda.\forall\mu.(\phi[\lambda] \wedge \phi[\mu] \Rightarrow \phi[g(\lambda,\mu)])\end{array}}{\forall\upsilon.\phi[\upsilon]}$$

# Objects and Base Cases

Object constants: $a$, $b$

Binary function constants: $f$, $g$

$$\phi[a]$$

$$\phi[b]$$

$$\forall\lambda.\forall\mu.(\phi[\lambda] \wedge \phi[\mu] \Rightarrow \phi[f(\lambda,\mu)])$$

$$\forall\lambda.\forall\mu.(\phi[\lambda] \wedge \phi[\mu] \Rightarrow \phi[g(\lambda,\mu)])$$

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$$\forall\upsilon.\phi[\upsilon]$$

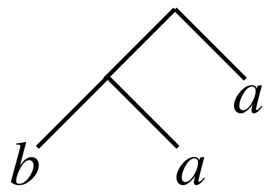
# Functions and Inductive Cases

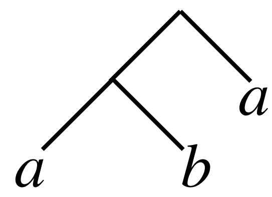
Object constants: $a$, $b$

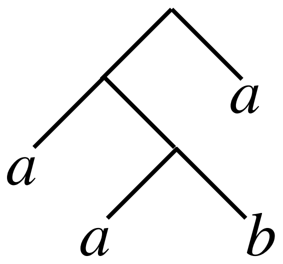
Binary function constants: $f$, $g$

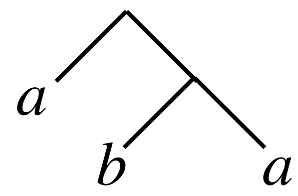
$$\frac{\begin{array}{l}\phi[a] \\ \phi[b] \\ \forall\lambda.\forall\mu.(\phi[\lambda] \land \phi[\mu] \Rightarrow \phi[f(\lambda,\mu)]) \\ \forall\lambda.\forall\mu.(\phi[\lambda] \land \phi[\mu] \Rightarrow \phi[g(\lambda,\mu)])\end{array}}{\forall\upsilon.\phi[\upsilon]}$$

# Trees



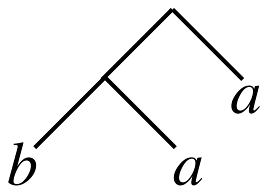



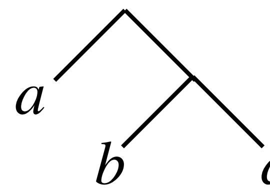



# Tree Vocabulary

Object constants: *a*, *b*
Unary function constants: *cons*




*cons*(*cons*(*b*,*a*),*a*)



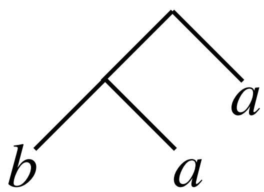

*cons*(*a*,*cons*(*b*,*a*))

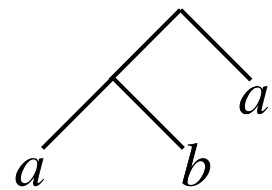
Unary relation constants: *symmetric*, *uniform*
Binary relation constant: *subtree*, ***congruent***

# Congruence

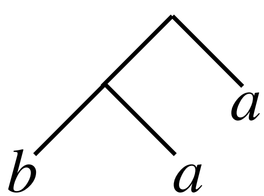
Two trees are *congruent* if and only if they have the *same shape*. (Labels on leaf nodes irrelevant.)

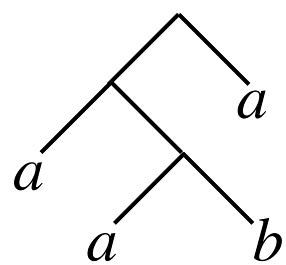
Examples:

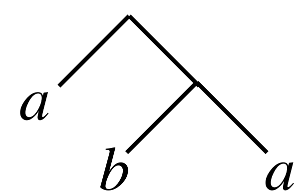



Non-Examples:



# Definition

Congruence of "atomic" trees

$$
\begin{gathered}
congruent(a, a) \\
congruent(a, b) \\
congruent(b, a) \\
congruent(b, b)
\end{gathered}
$$

Congruence of "compound" trees:

$$
\begin{aligned}
\forall u.\forall v.\forall x.\forall y.(&congruent(cons(u, v), cons(x, y)) \Leftrightarrow \\
&congruent(u, x) \ \wedge congruent(v, y))
\end{aligned}
$$

Non-Congruence of mixed trees:

$$
\forall x.\forall y.(\neg congruent(a, cons(x, y)) \wedge \neg congruent(cons(x, y), a))
$$

$$
\forall x.\forall y.(\neg congruent(b, cons(x, y)) \wedge \neg congruent(cons(x, y), b))
$$

# Properties

Congruence is an equivalence relation.

Reflexivity

$$\forall x.congruent(x, x)$$

Symmetry

$$\forall x.\forall y.(congruent(x, y) \Rightarrow congruent(y, x))$$

Transitivity

$$\forall x.\forall y.\forall z.(congruent(x, y) \land congruent(y, z) \Rightarrow congruent(x, z))$$

# Problem

Given:

$$congruent(a, a)$$
$$congruent(a, b)$$
$$congruent(b, a)$$
$$congruent(b, b)$$

$$\forall u.\forall v.\forall x.\forall y.(congruent(cons(u, v), cons(x, y)) \Leftrightarrow congruent(u, x) \ \wedge congruent(v, y))$$

Prove:

$$\forall x.congruent(x, x)$$

# Problem Restated using ≅

Given:

$$a \cong a$$
$$a \cong b$$
$$b \cong a$$
$$b \cong b$$

$$\forall u.\forall v.\forall x.\forall y.(cons(u, v) \cong cons(x, y) \Leftrightarrow u \cong x \ \wedge \ v \cong y)$$

Prove:

$$\forall x.x \cong x$$

# Induction Strategy

Our goal is to prove that congruence is reflexive.

$$\forall x. x \cong x$$

Base Cases:

$$a \cong a$$
$$b \cong b$$

Inductive Case:

$$\forall x. \forall y. (x \cong x \land y \cong y \Rightarrow cons(x, y) \cong cons(x, y))$$

# Proof of Reflexivity

| | | |
|---|---|---|
| 1. | $a \cong a$ | Premise |
| 2. | $a \cong b$ | Premise |
| 3. | $b \cong a$ | Premise |
| 4. | $b \cong b$ | Premise |
| 5. | $\forall u.\forall v.\forall x.\forall y.(cons(u, v) \cong cons(x, y) \Leftrightarrow u \cong x \wedge v \cong y)$ | Premise |

# Proof of Reflexivity

| | | |
|---|---|---|
| 1. | $a \cong a$ | Premise |
| 2. | $a \cong b$ | Premise |
| 3. | $b \cong a$ | Premise |
| 4. | $b \cong b$ | Premise |
| 5. | $\forall u.\forall v.\forall x.\forall y.(cons(u, v) \cong cons(x, y) \Leftrightarrow u \cong x \wedge v \cong y)$ | Premise |
| 6. | $cons(c, d) \cong cons(c, d) \Leftrightarrow c \cong c \wedge d \cong d$ | 4 x UE: 5 |
| 7. | $c \cong c \wedge d \cong d \Rightarrow cons(c, d) \cong cons(c, d)$ | BE: 6 |

# Proof of Reflexivity

| | | |
|---|---|---|
| 1. | $a \cong a$ | Premise |
| 2. | $a \cong b$ | Premise |
| 3. | $b \cong a$ | Premise |
| 4. | $b \cong b$ | Premise |
| 5. | $\forall u.\forall v.\forall x.\forall y.(cons(u, v) \cong cons(x, y) \Leftrightarrow u \cong x \wedge v \cong y)$ | Premise |
| 6. | $cons(c, d) \cong cons(c, d) \Leftrightarrow c \cong c \wedge d \cong d$ | 4 x UE: 5 |
| 7. | $c \cong c \wedge d \cong d \Rightarrow cons(c, d) \cong cons(c, d)$ | BE: 6 |
| 8. | $\forall x.\forall y.(x \cong x \wedge y \cong y \Rightarrow cons(x, y) \cong cons(x, y))$ | 2 x UI: 7 |

# Proof of Reflexivity

| | | |
|---|---|---|
| 1. | $a \cong a$ | Premise |
| 2. | $a \cong b$ | Premise |
| 3. | $b \cong a$ | Premise |
| 4. | $b \cong b$ | Premise |
| 5. | $\forall u.\forall v.\forall x.\forall y.(cons(u, v)\cong cons(x, y) \Leftrightarrow u\cong x \wedge v\cong y)$ | Premise |
| 6. | $cons(c, d)\cong cons(c, d) \Leftrightarrow c\cong c \wedge d\cong d$ | 4 x UE: 5 |
| 7. | $c\cong c \wedge d\cong d \Rightarrow cons(c, d)\cong cons(c, d)$ | BE: 6 |
| 8. | $\forall x.\forall y.(x\cong x \wedge y\cong y \Rightarrow cons(x, y)\cong cons(x, y))$ | 2 x UI: 7 |
| 9. | $\forall x.(x \cong x)$ | Ind: 1, 4, 8 |

# Analysis

# Logical Entailment and Provability

*Logical Entailment*: A set of premises Δ *logically entails* a conclusion φ ($\Delta \models \varphi$) if and only if every truth assignment that satisfies Δ also satisfies φ.

*Provability*: If there exists a proof of a sentence φ from a set Δ of premises using the rules of inference in R, we say that φ is *provable* from Δ using R (written $\Delta \vdash_R \varphi$).

# Soundness and Completeness

A proof system is *sound* if and only if every provable conclusion is logically entailed.

$$\text{If } \Delta \vdash \varphi, \text{ then } \Delta \vDash \varphi.$$

A proof system is *complete* if and only if every logical conclusion is provable.

$$\text{If } \Delta \vDash \varphi, \text{ then } \Delta \vdash \varphi.$$

# Fitch for Functional Logic

Theorem: Fitch is sound and complete for **Relational Logic**.

$\Delta \models \varphi$ if and only if $\Delta \vdash_{\text{Fitch}} \varphi$.

Theorem: Fitch (with induction) is *sound* for **Functional Logic** but, unfortunately, it is *not complete*.

If $\Delta \vdash_{\text{Fitch}} \varphi$, then $\Delta \models \varphi$
but the converse is *not necessarily* true.

# We Just Need To Try Harder, Right?

Question: *Is there some proof system that allows us to prove all logically entailed sentences in Functional Logic?*

Answer: No.

# Why Not?

*Unfortunately, there is no proof system that allows us to prove all logically entailed sentences in Functional Logic.*

Undecidable Problems:

- Solvability of arbitrary Diophantine equations
- Halting problem for Turing machines

We can encode problems like these as questions of logical entailment in Functional Logic. If logical entailment for FL were decidable, these problems would be decidable.

# Compactness

A logic is *compact* if and only if every unsatisfiable set of sentences has a finite subset that is unsatisfiable.

Suppose $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, ...\}$ is unsatisfiable.
Then, some finite subset $\{\varphi_2, \varphi_7, , ..., \varphi_{201}, \varphi_{878}\}$ is unsatisfiable.

*Theorem*: Propositional Logic is compact.
*Theorem*: Relational Logic is compact.

# Significance

Theorem: $\Delta \models \varphi$ if and only if $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Upshot: Given an infinite set of sentences, we can enumerate finite subsets and check for unsatisfiability. If we find one, logical entailment holds.

# Functional Logic is *not* Compact

*Theorem*: Functional Logic is *not* compact.

Proof: There is an infinite set of sentences that is unsatisfiable and yet every finite subset is satisfiable.

$$\{p(0), p(s(0)), p(s(s(0))), \ldots, \exists x.\neg p(x)\}$$

Significance: Some conclusions in Functional Logic have only *infinite* proofs.