Introduction to Logic
Natural Deduction

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Proof Systems

Popular Types of Proof Systems:

- Direct Proofs (Hilbert)
- → Natural Deduction (Fitch)
- Refutation proofs (Resolution / Robinson)

Others:

- Gentzen Systems
- Sequent Calculi
- and so forth

Example - Transitivity Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$	IC
4.	$(p \Rightarrow (q \Rightarrow r))$	IE: 2, 3
5.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	ID
6.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	IE: 4, 5
7.	$p \Rightarrow r$	IE: 1, 6

How do we choose instances of IC and ID?

Transitivity - Related Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$ and p, prove r.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	p	Premise
4.	q	IE: 1, 3
5.	r	IE: 2, 4

Deduction Theorem

$\Delta \vdash (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \vdash \psi$

Deduction Theorem

Deduction Theorem: $\Delta \vdash (\varphi \Rightarrow \psi)$ if and only if $\Delta \cup \{\varphi\} \vdash \psi$.

$$\{(p \Rightarrow q), (q \Rightarrow r)\} \quad \vdash \quad (p \Rightarrow r)$$

if and only if

$$\{(p \Rightarrow q), (q \Rightarrow r), p\} \quad \vdash \quad r$$

There is a proof of $(p \Rightarrow r)$ from $\{(p \Rightarrow q), (q \Rightarrow r)\}$ if and only if there is a proof of r from $\{(p \Rightarrow q), (q \Rightarrow r), p\}$.

Transitivity - Hilbert Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$, prove $(p \Rightarrow r)$.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$	IC
4.	$(p \Rightarrow (q \Rightarrow r))$	IE: 2, 3
5.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	ID
6.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	IE: 4, 5
7.	$p \Rightarrow r$	IE: 1, 6

NB: This is a proof of $(p \Rightarrow r)$ from $(p \Rightarrow q)$ and $(q \Rightarrow r)$.

Transitivity - Related Proof

Given $(p \Rightarrow q)$ and $(q \Rightarrow r)$ and p, prove r.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	p	Premise
4.	q	IE: 1, 3
5.	r	IE: 2, 4

*NB: This is **not** itself a proof of $(p \Rightarrow r)$ from $(p \Rightarrow q)$ and $(q \Rightarrow r)$, but the deduction theorem tells us there **is** such a proof.*

Natural Deduction

Natural Deduction

Natural Deduction incorporates the deduction theorem in a *new type of inference rule* and an *extended notion of proof*.

More "natural" for most people.

No need for axiom schemata!!

Essence of Natural Deduction

Making Assumptions

e.g. assume p

Using Assumptions

e.g. derive q

Discharging Assumptions leading to implications

e.g. conclude $p \Rightarrow q$

Making Assumptions

In a natural deduction proof, it is permissible to make an arbitrary *assumption* in a nested proof.

NB: We can assume ***anything we like***. We can assume ***anything we like***. We can assume ***anything we like***.

p

$p \Rightarrow q$

$p \lor \neg p$

$p \land \neg p$

This is **okay** since (1) an assumption is used only within a subproof and (2) everything we prove in that subproof **depends** on that assumption (as we shall see in a moment).

Using Assumptions

Such assumptions can be used *within* **subproofs** to derive conclusions using ordinary rules of inference.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$\mid p$	Assumption
4.	$\mid q$	Implication Elimination: 1, 3
5.	$\mid r$	Implication Elimination: 2, 4

NB: This is a proof of r *under the assumption* that p is true. It is *not* a proof of r from the premises alone.

Discharging Assumptions

After exiting a subproof, we infer an **implication** with the assumption as antecedent and the conclusion as consequent.

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$\quad \mid p$	Assumption
4.	$\quad \mid q$	Implication Elimination: 1, 3
5.	$\quad \mid r$	Implication Elimination: 2, 4
6.	$p \Rightarrow r$	Implication Introduction: 3, 5

Everything we prove in a subproof depends on the assumption, and we capture that dependence in the derived implication.

Our Transitivity Example

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	$\quad p$	Assumption
4.	$\quad q$	Implication Elimination: 3, 1
5.	$\quad r$	Implication Elimination: 4, 2
6.	$p \Rightarrow r$	Implication Introduction: 3, 5

Caveat

An ordinary rule of inference *applies* to a subproof at any level of nesting if and only if there is an instance of the rule in which all of the premises occur earlier in the subproof or in a superproof of that subproof.

Importantly, it is *not* permissible to apply an ordinary rule of inference to items that occur in other subproofs.

Bad Proof

1.	$p \Rightarrow q$	Premise
2.	$q \Rightarrow r$	Premise
3.	p	Assumption
4.	q	Implication Elmination: 1, 3
5.	r	Implication Elmination: 2, 4
6.	$p \Rightarrow r$	Implication Introduction: 3, 5
7.	$\neg r$	Assumption
X 8.	r	Implication Elmination: 2, 4 X
9.	$\neg r \Rightarrow r$	Implication Introduction: 7, 8

Bad Proof

	1.	$p \Rightarrow q$	Premise
	2.	$q \Rightarrow r$	Premise
	3.	p	Assumption
	4.	q	Implication Elimination: 1, 3
	5.	r	Implication Elmination: 2, 4
	6.	$p \Rightarrow r$	Implication Introduction: 3, 5
X	7.	r	Implication Elimination: 2, 4 X

Terminology

A *structured rule of inference* is a pattern of reasoning consisting of one or more schemas, called *premises*, and one or more additional schemas, called *conclusions*, in which one of the premises is a condition of the form $\phi \vdash \psi$.

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Translation: If there is a subproof with assumption ϕ and conclusion ψ, then we conclude $\phi \Rightarrow \psi$ outside the subproof.

This new rule of inference is called *Implication Introduction*.

Terminology

A *structured proof* of a conclusion from a set of premises is a sequence of (possibly nested) sentences terminating in an occurrence of the conclusion at the top level of the proof. Each step in the proof must be either

(1) a premise (at the top level),

(2) an assumption, or

(3) the result of applying an ordinary rule of inference *or* a structured rule of inference to earlier items in the sequence.

Fitch

Implications

Implication Introduction (II):

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

Implication Elimination (IE):

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}$$

Negations

Negation Introduction (NI):

$$\frac{\begin{array}{l}\phi \Rightarrow \chi \\ \phi \Rightarrow \neg\chi\end{array}}{\neg\phi}$$

Negation Elimination (NE):

$$\frac{\neg\neg\phi}{\phi}$$

Conjunctions

And Introduction (AI):

$$\frac{\begin{array}{l}\phi \\ \psi\end{array}}{\phi \land \psi}$$

And Elimination (AE):

$$\frac{\phi \land \psi}{\begin{array}{l}\phi \\ \psi\end{array}}$$

Disjunctions

Or Introduction (OI):

$$\frac{\phi}{\phi \vee \psi}$$

Or Elimination (OE):

$$\frac{\begin{array}{l}\phi \vee \psi \\ \phi \Rightarrow \chi \\ \psi \Rightarrow \chi\end{array}}{\chi}$$

Equivalences / Biconditionals

Biconditional Introduction (BI):

$$\frac{\phi \Rightarrow \psi \qquad \psi \Rightarrow \phi}{\phi \Leftrightarrow \psi}$$

Biconditional Elimination (BE):

$$\frac{\phi \Leftrightarrow \psi}{\phi \Rightarrow \psi \qquad \psi \Rightarrow \phi}$$

Examples

Mary and Pat and Quincy

If Mary loves Pat, then Mary loves Quincy. If it is Monday, then Mary loves Pat or Quincy. If it is Monday, does Mary love Quincy?

Premises:

$$p \Rightarrow q$$

$$m \Rightarrow p \vee q$$

Goal:

$$m \Rightarrow q$$

Proof

Goal: $m \Rightarrow q$

1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise

Proof

Goal: $m \Rightarrow q$

1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\mid m$	Assumption

Proof

Goal: $m \Rightarrow q$

1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\quad m$	Assumption
4.	$\quad p \vee q$	Implication Elimination: 2, 3

Proof

Goal: $m \Rightarrow q$

1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\quad m$	Assumption
4.	$\quad p \vee q$	Implication Elimination: 2, 3
5.	$\quad\quad q$	Assumption
6.	$\quad q \Rightarrow q$	Implication Introduction: 5, 5

Proof

Goal: $m \Rightarrow q$

1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\quad m$	Assumption
4.	$\quad p \vee q$	Implication Elimination: 2, 3
5.	$\quad\quad q$	Assumption
6.	$\quad q \Rightarrow q$	Implication Introduction: 5, 5
7.	$\quad q$	Or Elimination: 4, 1, 6

Proof

Goal: $m \Rightarrow q$

1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\quad m$	Assumption
4.	$\quad p \vee q$	Implication Elimination: 2, 3
5.	$\quad\quad q$	Assumption
6.	$\quad q \Rightarrow q$	Implication Introduction: 5, 5
7.	$\quad q$	Or Elimination: 4, 1, 6
8.	$m \Rightarrow q$	Implication Introduction: 3, 7

Reflexivity - Hilbert Proof

Prove $(p \Rightarrow p)$.

1.	$p \Rightarrow (p \Rightarrow p)$	IC
2.	$p \Rightarrow ((p \Rightarrow p) \Rightarrow p)$	IC
3.	$p \Rightarrow ((p \Rightarrow p) \Rightarrow p) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))$	ID
4	$(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$	IE: $2, 3$
5.	$(p \Rightarrow p)$	IE: $1, 4$

Reflexivity - Fitch Proof

Prove $(p \Rightarrow p)$.

1.	$\mid p$	Assumption
2.	$p \Rightarrow p$	Implication Introduction: 1, 1

Negation Elimination - Hilbert Proof

Prove $(\neg\neg p \Rightarrow p)$.

1	$\neg\neg p \Rightarrow (\neg\neg\neg\neg p \Rightarrow \neg\neg p)$	IC
2.	$(\neg\neg\neg\neg p \Rightarrow \neg\neg p) \Rightarrow (\neg p \Rightarrow \neg\neg\neg p)$	IR
3.	$\neg\neg p \Rightarrow (\neg p \Rightarrow \neg\neg\neg p)$	Transitivity: $1, 2$
4.	$(\neg p \Rightarrow \neg\neg\neg p) \Rightarrow (\neg\neg p \Rightarrow p)$	IR
5.	$\neg\neg p \Rightarrow (\neg\neg p \Rightarrow p)$	Transitivity: $3, 4$
6.	$(\neg\neg p \Rightarrow (\neg\neg p \Rightarrow p)) \Rightarrow$ $((\neg\neg p \Rightarrow \neg\neg p) \Rightarrow (\neg\neg p \Rightarrow p))$	ID
7.	$(\neg\neg p \Rightarrow \neg\neg p) \Rightarrow (\neg\neg p \Rightarrow p)$	IE: $6, 5$
8.	$\neg\neg p \Rightarrow \neg\neg p$	Reflexivity
9.	$\neg\neg p \Rightarrow p$	IE: $7, 8$

Negation Elimination - Fitch Proof

Prove $(\neg\neg p \Rightarrow p)$.

1	$\mid \neg\neg p$	Assumption
2.	$\mid p$	Negation Elimination: 1
3.	$\neg\neg p \Rightarrow p$	Implication Introduction: 1, 2

Soundness and Completeness

Logical Entailment and Provability

A set of premises Δ *logically entails* a conclusion φ ($\Delta \models \varphi$) if and only if every interpretation that satisfies Δ also satisfies φ.

If there exists a proof of a sentence ϕ from a set Δ of premises using the rules of inference in R, we say that ϕ is *provable* from Δ using R (written $\Delta \vdash_R \phi$).

Soundness and Completeness

A proof system is *sound* if and only if every provable conclusion is logically entailed.

$$\text{If } \Delta \vdash \phi, \text{ then } \Delta \models \phi.$$

A proof system is *complete* if and only if every logically entailed conclusion is provable.

$$\text{If } \Delta \models \phi, \text{ then } \Delta \vdash \phi.$$

Fitch

Theorem: Fitch is sound and complete for Propositional Logic.

$$\Delta \models \varphi \text{ if and only if } \Delta \vdash_{\text{Fitch}} \phi.$$

Upshot: The truth table method and the Fitch method succeed in exactly the same cases!

Comparative Power

*Fitch can do anything Hilbert can do!**+#

* *No axiom schemata necessary.*

\+ *More intuitive than Hilbert.*

\# *Proofs are usually shorter.*

Implication Creation

Prove $(p \Rightarrow (q \Rightarrow p))$.

1.	p	Assumption
2.	q	Assumption
3	p	Reiteration
4	$q \Rightarrow p$	II: 2, 3
5.	$p \Rightarrow (q \Rightarrow p)$	II: 1, 4

Implication Distribution

Prove $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$p \Rightarrow q$	Assumption
3	p	Assumption
4	q	IE: 2, 3
5.	$q \Rightarrow r$	IE: 1, 3
6.	r	IE: 5, 4
7.	$p \Rightarrow r$	II: 3, 6
8.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	II: 2, 7
9.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	II: 1, 8

Implication Reversal

Prove $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$.

1.	$\neg q \Rightarrow \neg p$	Assumption
2.	p	Assumption
3	$\neg q$	Assumption
4	p	Reiteration
5.	$\neg q \Rightarrow p$	II: 3, 4
6.	$\neg\neg q$	NI: 5, 1
7.	q	NE: 6
8.	$p \Rightarrow q$	II: 2, 7
9.	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$	II: 1, 8

Practical Matters

Theorem Proving Requires Search

Good News: There *is* an algorithm for determining logical entailment in Propositional Logic (using Truth tables).

Bad News: Truth tables can be very large.

Good News: Proofs, once found, are usually smaller than truth tables. Moreover, there *are* algorithms for finding Propositional Logic proofs using Hilbert and Fitch.

Bad News: These algorithms can be expensive. (In fact, the worst case is no better than the truth table method!)
Theorem proving requires search.

Good News: In many cases, proofs can be found quickly.

Reasoning Tips

Tip 1: Given a *goal* ($\phi \Rightarrow \psi$), it is often good to assume ϕ, prove ψ, and then use Implication Introduction.

1.	q	Premise
2.	$\quad p$	Assumption
3.	$\quad q$	Reiteration: 1
4.	$p \Rightarrow q$	II: 2, 3

Reasoning Tips

Tip 2: Given a *goal* **(ϕ ∧ ψ)**, prove ϕ, prove ψ, and then use And Introduction to derive (ϕ ∧ ψ).

1.	$p \Rightarrow q$	Premise
2.	$p \Rightarrow r$	Premise
3.	p	Premise
4.	q	IE: 1, 3
5.	r	IE: 2, 3
6.	$q \wedge r$	AI: 4, 5

Reasoning Tips

Tip 3: Given a *goal* **($\phi \vee \psi$)**, try to prove ϕ *or* prove ψ (only one is needed), then use Or Introduction to disjoin with *anything else*.

1.	$p \wedge q$	Premise
2.	p	AE: 1
3.	q	AE: 1
4.	$q \vee r$	OI: 3

Reasoning Tips

Tip 4: Given a *goal* of the form **¬ϕ**, assume ϕ and derive the sentence (ϕ ⇒ ψ), assume ϕ again and derive the sentence ¬ψ leading to (ϕ ⇒ ¬ψ), and use Negation Introduction to derive ¬ϕ.

1.	$p \Rightarrow q$	Premise
2.	$\neg q$	Premise
3.	| p	Assumption
4.	| $\neg q$	Reiteration: 2
5.	$p \Rightarrow \neg q$	II: 3, 4
6.	$\neg p$	NI: 1, 5

Reasoning Tips

Tip 5: Given a *goal* ϕ, assume $\neg\phi$, prove a contradiction, thereby deriving $\neg\neg\phi$, and then apply Negation Elimination to get ϕ.

1.	$\neg p \Rightarrow q$	Premise
2.	$\neg q$	Premise
3.	$\quad \mid \neg p$	Assumption
4.	$\quad \mid \neg q$	Reiteration: 2
5.	$\neg p \Rightarrow \neg q$	II: 3, 4
6.	$\neg\neg p$	NI: 1, 5
7.	p	NE: 6

Reasoning Tips

Tip 6: Given a *premise* of the form **($\phi \Rightarrow \psi$)** and a goal **ψ**, try proving ϕ and then use Implication Elimination to derive ψ.

1.	$p \wedge q \Rightarrow r$	Premise
2.	p	Premise
3.	q	Premise
4.	$p \wedge q$	AI: 2 3
5.	r	IE: 1, 4

Reasoning Tips

Tip 7: Given a *premise* **(ϕ ∨ ψ)** and a goal **χ**, try proving (ϕ ⇒ χ) and (ψ ⇒ χ) and use Or Elimination to derive χ.

1.	$p \vee q$	Premise
2.	$p \Rightarrow q$	Premise
3.	$\mid q$	Assumption
4.	$q \Rightarrow q$	II: 3, 3
5.	q	OE: 1, 2, 4

Reasoning Tips

Tip 8: Given a *premise* **(ϕ ∧ ψ)**, consider splitting it into its constituent conjuncts.

1.	$p \wedge q$	Premise
2.	p	AE: 1
3.	q	AE: 1
4.	$q \vee r$	OI: 3

Fitch Online System

Course Website

http://logica.stanford.edu

Example

1.	$p \Rightarrow q$	Premise
2.	$m \Rightarrow p \vee q$	Premise
3.	$\quad m$	Assumption
4.	$\quad p \vee q$	Implication Elimination: 2, 3
5.	$\quad\quad q$	Assumption
6.	$\quad q \Rightarrow q$	Implication Introduction: 5, 5
7.	$\quad q$	Or Elimination: 4, 1, 6
8.	$m \Rightarrow q$	Implication Introduction: 3, 7

Example

1.	$p \Rightarrow r$	Premise
2.	$q \Rightarrow s$	Premise
3.	$p \vee q$	Premise
4.	$\quad\mid\ p$	Assumption
5.	$\quad\mid\ r$	Implication Elimination: 1, 4
6.	$\quad\mid\ r \vee s$	Or Introduction: 5
7.	$p \Rightarrow r \vee s$	Implication Introduction: 4, 6
8.	$\quad\mid\ q$	Assumption
9.	$\quad\mid\ s$	Implication Elimination: 2, 8
10.	$\quad\mid\ r \vee s$	Or Introduction: 9
11.	$q \Rightarrow r \vee s$	Implication Introduction: 8, 10
12.	$r \vee s$	Or Elimination: 3, 7, 11

Example

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4	$y \Leftrightarrow \neg m$	Premise

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	$\mid t$	Assumption

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	$\mid t$	Assumption
8.	$\mid \neg m$	IE: 2, 7

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	$\quad$ | t	Assumption
8.	$\quad$ | $\neg m$	IE: 2, 7
9.	$\quad$ | y	IE: 6, 8

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	| t	Assumption
8.	| $\neg m$	IE: 2, 7
9.	| y	IE: 6, 8
10.	$t \Rightarrow y$	II: 7, 9

Coin Game

Heads, you win. Tails, I lose. Prove that you win.

1	$h \Rightarrow y$	Premise
2.	$t \Rightarrow \neg m$	Premise
3.	$h \vee t$	Premise
4	$y \Leftrightarrow \neg m$	Premise
5.	$y \Rightarrow \neg m$	BE: 4
6.	$\neg m \Rightarrow y$	BE: 4
7.	| t	Assumption
8.	| $\neg m$	IE: 2, 7
9.	| y	IE: 6, 8
10.	$t \Rightarrow y$	II: 7, 9
11.	y	OE: 3, 1, 10