Introduction to Logic

Natural Deduction

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Given \((p \Rightarrow q)\) and \((q \Rightarrow r)\), prove \((p \Rightarrow r)\).

1. \(p \Rightarrow q\)  
2. \(q \Rightarrow r\)  
3. \((q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))\)  
4. \((p \Rightarrow (q \Rightarrow r))\)  
5. \((p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))\)  
6. \((p \Rightarrow q) \Rightarrow (p \Rightarrow r)\)  
7. \(p \Rightarrow r\)
Structured Proofs

1. \( p \Rightarrow q \) Premise
2. \( q \Rightarrow r \) Premise
3. \( p \) Assumption
4. \( q \) Implication Elimination: 3, 1
5. \( r \) Implication Elimination: 4, 2
6. \( p \Rightarrow r \) Implication Introduction: 3, 5
Natural Deduction

Making Assumptions
  e.g. assume $p$

Applying Ordinary Rules of Inference to derive conclusions
  e.g. derive $q$

Discharging Assumptions leading to implications
  e.g. conclude $p \Rightarrow q$
Conditional Proofs
In a conditional proof, it is permissible to make an arbitrary assumption or hypothetical in a nested proof. The assumption need not be in the original premise set.

Such assumptions can be used within the nested proof. However, they may not be used outside of the subproof in which they appear.
Example

1. $p \Rightarrow q$  Premise
2. $q \Rightarrow r$  Premise
3. $p$  Assumption
4. $q$  Implication Elimination: 3, 1
5. $r$  Implication Elimination: 4, 2
6. $p \Rightarrow r$  Implication Introduction: 3, 5
An ordinary rule of inference *applies* to a proof at any level of nesting if and only there is an instance of the rule in which all of the premises occur earlier in the nested proof or in some “superproof” of the nested proof.

Importantly, it is *not* permissible to apply an ordinary rule of inference to premises in subproofs of a nested proof or in other subproofs of a superproof of a nested proof.
Example

1. \( p \Rightarrow q \)  Premise

2. \( q \Rightarrow r \)  Premise

3. \( p \)  Assumption

4. \( q \)  Implication Elimination: 3, 1

5. \( r \)  Implication Elimination: 4, 2

6. \( p \Rightarrow r \)  Implication Introduction: 3, 5
Bad Proof

1. \( p \Rightarrow q \) Premise
2. \( q \Rightarrow r \) Premise
3. \( p \) Assumption
4. \( q \) Implication Elimination: 1, 3
5. \( r \) Implication Elimination: 2, 4
6. \( p \Rightarrow r \) Implication Introduction: 3, 5
7. \( r \) Implication Elimination: 2, 4
1. \( p \implies q \) Premise
2. \( q \implies r \) Premise
3. \( p \) Assumption
4. \( q \) Implication Elimination: 1, 3
5. \( r \) Implication Elimination: 2, 4
6. \( p \implies r \) Implication Introduction: 3, 5
7. \( \neg r \) Assumption
8. \( r \) Implication Elimination: 2, 4
9. \( \neg r \implies r \) Implication Introduction: 7, 8
Structured Rules of Inference

A *structured rule of inference* is a pattern of reasoning consisting of one or more schemas, called *premises*, and one or more additional schemas, called *conclusions*, in which one of the premises is a condition of the form $\phi \vdash \psi$.

\[
\frac{}{\phi \vdash \psi} \\
\phi \Rightarrow \psi
\]

This schema is called *Implication Introduction*. 
A structured rule of inference *applies* to a nested proof if and only if there is an instance of the rule in which all of the premises are satisfied.

A premise that is an ordinary schema is satisfied if and only if it occurs earlier in the nested proof or in any “superproof” of that nested subproof.

A premise of the form $\phi \vdash \psi$ is satisfied if and only if the nested proof has $\phi$ as an assumption and terminates in $\psi$. 
A *structured proof* of a conclusion from a set of premises is a sequence of (possibly nested) sentences terminating in an occurrence of the conclusion at the top level of the proof. Each step in the proof must be either (1) a premise (at the top level), (2) an assumption, or (3) the result of applying an ordinary or structured rule of inference to earlier items in the sequence (subject to the constraints given above).
Fitch
Negation Introduction (NI):

\[ \phi \Rightarrow \chi \]
\[ \phi \Rightarrow \neg \chi \]
\[ \neg \phi \]

Negation Elimination (NE):

\[ \neg \neg \phi \]
\[ \phi \]
And Introduction (AI):

\[ \phi \\
\psi \\
\hline
\phi \land \psi \]

And Elimination (AE):

\[ \phi \land \psi \\
\phi \\
\psi \]
Or Introduction (BI):

\[ \phi \quad \frac{\phi \lor \psi}{\phi \lor \psi} \]

Or Elimination (BE):

\[ \phi \lor \psi \quad \phi \Rightarrow \chi \quad \psi \Rightarrow \chi \quad \frac{\chi}{\chi} \]
And Introduction (AI):

\[ \frac{\phi \vdash \psi}{\phi \Rightarrow \psi} \]

Implications Elimination (AE):

\[ \frac{\phi \Rightarrow \psi}{\phi \quad \psi} \]
Biconditional Introduction (BI):

\[ \phi \iff \psi \]

\[ \psi \iff \phi \]

\[ \phi \iff \psi \]

Biconditional Elimination (BE):

\[ \phi \iff \psi \]

\[ \phi \Rightarrow \psi \]

\[ \psi \Rightarrow \phi \]
Transitivity - Hilbert Proof

Given \((p \Rightarrow q)\) and \((q \Rightarrow r)\), prove \((p \Rightarrow r)\).

1. \(p \Rightarrow q\)  
   Premise
2. \(q \Rightarrow r\)  
   Premise
3. \((q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))\)  
   IC
4. \((p \Rightarrow (q \Rightarrow r))\)  
   IE: 2, 3
5. \((p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))\)  
   ID
6. \((p \Rightarrow q) \Rightarrow (p \Rightarrow r)\)  
   IE: 4, 5
7. \(p \Rightarrow r\)  
   IE: 1, 6
Given \((p \Rightarrow q)\) and \((q \Rightarrow r)\), prove \((p \Rightarrow r)\).

1. \(p \Rightarrow q\) Premise
2. \(q \Rightarrow r\) Premise
3. \(p\) Assumption
4. \(q\) Implication Elimination: 3, 1
5. \(r\) Implication Elimination: 4, 2
6. \(p \Rightarrow r\) Implication Introduction: 3, 5
Prove \((p \Rightarrow p)\).

1. \(p \Rightarrow (p \Rightarrow p)\) 
   \hspace*{2cm} IC
2. \(p \Rightarrow ((p \Rightarrow p) \Rightarrow p)\) 
   \hspace*{2cm} IC
3. \(p \Rightarrow ((p \Rightarrow p) \Rightarrow p) \Rightarrow ((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p))\) 
   \hspace*{2cm} ID
4. \((p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)\) 
   \hspace*{2cm} IE: 2, 4
5. \((p \Rightarrow p)\) 
   \hspace*{2cm} IE: 1, 4
Prove \((p \Rightarrow p)\).

1. \(\text{\textbar} \quad p\) Assumption
2. \(p \Rightarrow p\) Implication Introduction: 1, 1
Given $p$ and $\neg p$, prove $q$.

1. $p$ \hspace{1cm} Premise
2. $\neg p$ \hspace{1cm} Premise
3. $\neg p \Rightarrow (\neg q \Rightarrow \neg p)$ \hspace{1cm} IC
4. $\neg q \Rightarrow \neg p$ \hspace{1cm} IE: 3, 2
5. $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$ \hspace{1cm} IR
6. $p \Rightarrow q$ \hspace{1cm} IE: 5, 4
7. $q$ \hspace{1cm} IE: 6, 1
Given $p$ and $\neg p$, prove $q$.

1. $p$  
   Premise
2. $\neg p$  
   Premise
3. $\neg q$  
   Assumption
4. $p$  
   Reiteration: 1
5. $\neg q \Rightarrow p$  
   Implication Introduction: 3, 4
6. $\neg q$  
   Assumption
7. $\neg p$  
   Reiteration: 2
8. $\neg q \Rightarrow \neg p$  
   Implication Introduction: 6, 7
9. $\neg\neg q$  
   Negation Introduction: 5, 8
10. $q$  
    Negation Elimination: 9
Negation Elimination - Hilbert Proof

Prove (¬¬p ⇒ p).

1. ¬¬p ⇒ (¬¬¬¬p ⇒ ¬¬p)  
   IC

2. (¬¬¬¬p ⇒ ¬¬p) ⇒ (¬p ⇒ ¬¬¬¬p)  
   IR

4. ¬¬p ⇒ (¬p ⇒ ¬¬¬¬p)  
   Transitivity: 1, 2

5. (¬p ⇒ ¬¬¬¬p) ⇒ (¬¬p ⇒ p)  
   IR

6. ¬¬p ⇒ (¬¬p ⇒ p)  
   Transitivity: 4, 5

7. (¬¬p ⇒ (¬¬p ⇒ p)) ⇒  
   ID

   ((¬¬p ⇒ ¬¬p) ⇒ (¬¬p ⇒ p))

8. (¬¬p ⇒ ¬¬p) ⇒ (¬¬p ⇒ p)  
   IE: 7, 6

9. ¬¬p ⇒ ¬¬p  
   Reflexivity

10. ¬¬p ⇒ p  
    IE: 8, 9
Prove ($\neg\neg p \Rightarrow p$).

1. $\neg\neg p$ Assumption
2. $p$ Negation Elimination: 1
4. $\neg\neg p \Rightarrow p$ Implication Introduction: 1, 2
Prove \((p \Rightarrow \neg \neg p)\).

1. \(p \Rightarrow p\)  Reflexivity
2. \(\neg p \Rightarrow \neg p\)  Reflexivity
3. \(p \Rightarrow \neg \neg p\)  Contradiction: 1, 2
Prove \((p \Rightarrow \neg \neg p)\).

1. \(\vert p\) Assumption
2. \(\vert \vert \neg p\) Assumption
3. \(\vert \vert p\) Reiteration: 1
4. \(\vert \neg p \Rightarrow p\) Implication Introduction: 2, 3
5. \(\vert \vert \neg p\) Assumption
6. \(\vert \neg p \Rightarrow \neg p\) Implication Introduction: 5, 5
7. \(\vert \neg \neg p\) Negation Introduction: 4, 6
8. \(p \Rightarrow \neg \neg p\) Implication Introduction: 1, 7
Soundness and Completeness
A set of premises $\Delta$ logically entails a conclusion $\varphi$ ($\Delta \models \varphi$) if and only if every interpretation that satisfies $\Delta$ also satisfies $\varphi$.

If there exists a proof of a sentence $\phi$ from a set $\Delta$ of premises using the rules of inference in R, we say that $\phi$ is provable from $\Delta$ using R (written $\Delta \vdash_R \phi$).
A proof system is *sound* if and only if every provable conclusion is logically entailed.

\[ \Delta \vdash \phi \implies \Delta \models \phi. \]

A proof system is *complete* if and only if every logical conclusion is provable.

\[ \Delta \models \phi \implies \Delta \vdash \phi. \]
Theorem: Fitch is sound and complete for Propositional Logic.

\[ \Delta \models \varphi \text{ if and only if } \Delta \vdash_{\text{Fitch}} \varphi. \]

Upshot: The truth table method and the proof method succeed in exactly the same cases!
Practical Matters
**Tip 1:** If the goal has the form \((\phi \Rightarrow \psi)\), it is often good to assume \(\phi\) and prove \(\psi\) and then use Implication Introduction to derive the goal.

1.  \(q\)  
   Premise
2.  \(p\)  
   Assumption
3.  \(q\)  
   Reiteration: 1
4.  \(p \Rightarrow q\)  
   IE: 2, 3
**Tip 2**: If the goal has the form \((\phi \land \psi)\), prove \(\phi\) and then prove \(\psi\) and then use And Introduction to derive \((\phi \land \psi)\).

**Tip 3**: If the goal has the form \((\phi \lor \psi)\), try to prove \(\phi\) or prove \(\psi\) (but we do not need to prove both), then use Or Introduction to disjoin with anything else.
Tip 4: If the goal has the form $\neg \phi$, (a) assume $\phi$ and derive the sentence $(\phi \Rightarrow \psi)$, (b) assume $\phi$ again and derive the sentence $\neg \psi$ leading to $(\phi \Rightarrow \neg \psi)$, and (c) use Negation Introduction to derive $\neg \phi$ as desired.

Tip 5: To prove any sentence $\phi$, assume $\neg \phi$, prove a contradiction as just discussed, thereby deriving $\neg \neg \phi$, and then apply Negation Elimination to get $\phi$. 
Tip 6: Given a premise of the form \((\phi \Rightarrow \psi)\) and a goal \(\psi\), try proving \(\phi\) and then use Implication Elimination to derive \(\psi\).

Tip 7: Given a premise \((\phi \lor \psi)\) and our goal is to prove \(\chi\), try proving \((\phi \Rightarrow \chi)\) and \((\psi \Rightarrow \chi)\) and use Or Elimination to derive \(\chi\).