Introduction to Logic
Propositional Analysis

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Syntax and Semantics

Syntax of Propositional Logic

$$\neg p$$
$$(p \wedge q)$$
$$(p \vee q)$$
$$(p \Rightarrow q)$$
$$(p \Leftrightarrow q)$$

Semantics of Propositional Logic

ϕ	$\neg\phi$
T	F
F	T

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Evaluation versus Satisfaction

Evaluation:

$$\begin{array}{lcl} p^i & = & \mathrm{T} \\ q^i & = & \mathrm{F} \end{array} \longrightarrow \begin{array}{lcl} (p \vee q)^i & = & \mathrm{T} \\ (\neg q)^i & = & \mathrm{T} \end{array}$$

Satisfaction:

$$\begin{array}{lcl} (p \vee q)^i & = & \mathrm{T} \\ (\neg q)^i & = & \mathrm{T} \end{array} \longrightarrow \begin{array}{lcl} p^i & = & \mathrm{T} \\ q^i & = & \mathrm{F} \end{array}$$

Programme for Today

Properties of Sentences

- Validity, Contingency, Unsatisfiability
- Satisfiability and Falsifiability

Relationships between Sentences

- Equivalence, Entailment, Consistency

Useful "Metatheorems"

- Equivalence, Unsatisfiability, Deduction, Consistency
- Substitution, Monotonicity, Ramification

Properties of Sentences

Truth Tables

A *truth table* is a table of all possible interpretations for the propositional constants in a language.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

One column per constant.

One row per interpretation.

For a language with n constants, there are 2^n interpretations.

Example

Constants			Premises
p	q	r	p & q => r
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Oddities

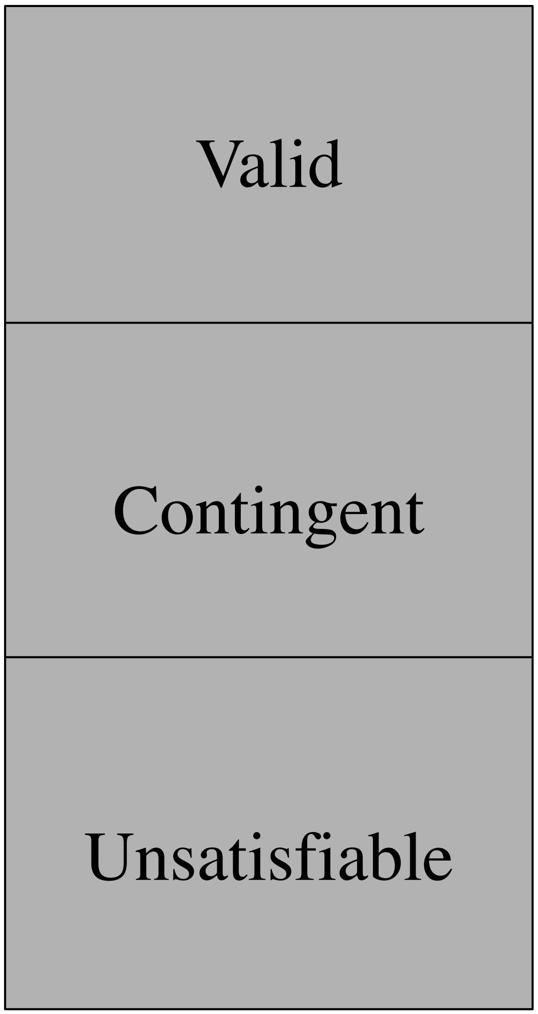
Constants			Premises
p	q	r	p & q => r | ~r
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Constants			Premises
p	q	r	p | (q | ~p) => r & ~r
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Properties of Sentences

Valid	A sentence is *valid* if and only if *every* interpretation satisfies it.
Contingent	A sentence is *contingent* if and only if *some* interpretation satisfies it and *some* interpretation falsifies it.
Unsatisfiable	A sentence is *unsatisfiable* if and only if *no* interpretation satisfies it.

Properties of Sentences



A sentences is *satisfiable* if and only if it is either valid or contingent.

A sentences is *falsifiable* if and only if it is contingent or unsatisfiable.

Possible Worlds

Constants			Premises		
p	q	r	p => q & r	q => r	~r
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	1	0
1	0	0	0	1	1
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	1	1	1

Possible Worlds

Constants			Premises		
p	**q**	**r**	**p => q & r**	**q => r**	**~r**
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	1	0
1	0	0	0	1	1
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	1	1	1

Possible Worlds

Constants			Premises		
p	**q**	**r**	**p => q & r**	**q => r**	**~r**
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	1	0
1	0	0	0	1	1
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	1	1	1

Possible Worlds

Constants			Premises		
p	q	r	p => q & r	q => r	~r
1	1	1	1	1	0
1	1	0	0	0	1
1	0	1	0	1	0
1	0	0	0	1	1
0	1	1	1	1	0
0	1	0	1	0	1
0	0	1	1	1	0
0	0	0	1	1	1

Possible Worlds

Constants			Premises
p	q	r	(p => q & r) | ~(p => q & r)
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

Possible Worlds

Constants			Premises
p	**q**	**r**	**(p => q & r) & ~(p => q & r)**
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

Valid Equivalences

Double Negation:

$$p \Leftrightarrow \neg\neg p$$

deMorgan's Laws:

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$

$$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$$

Implications:

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$$

Biconditionals:

$$(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (q \Rightarrow p))$$

Valid Implications

Implication Introduction:

$$p \Rightarrow (q \Rightarrow p)$$

Implication Distribution

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Implication Reversal

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

Relationships Between Sentences

Comparison of Sentences

Constants		Premises	
p	**q**	**p => q**	**~p | q**
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

Logical Equivalence

A sentence ϕ is *logically equivalent* to a sentence ψ if and only they have the same value for every propositional interpretation.

$(p \Rightarrow q)$ is logically equivalent to $(\neg p \vee q)$

p is logically equivalent to $\neg\neg p$

$(p \wedge q)$ is *not* logically equivalent to $(p \vee q)$

Another Comparison of Sentences

Constants		Premises	
p	**q**	**p & q**	**p | q**
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Logical Entailment

A premise φ *logically entails* a conclusion ψ (written as φ ⊨ ψ) if and only if every interpretation that satisfies φ also satisfies ψ.

$$(p \wedge q) \models (p \vee q)$$

$$p \models (p \vee q)$$

$$(p \wedge q) \models p$$

$$p \not\models (p \wedge q)$$

Logical Entailment ≠ Logical Equivalence

$$p \models (p \lor q)$$

$$(p \lor q) \not\models p$$

Analogy in arithmetic: inequalities rather than equations

Sets of Premises

A *set* of premises Δ *logically entails* a conclusion φ (written as Δ ⊨ φ) if and only if every interpretation that satisfies *all* of the premises also satisfies the conclusion.

$$\{p, q\} \models (p \land q)$$

Sets of Conclusions

A premise φ *logically entails* a *set* of conclusions if and only if every interpretation that satisfies the premise satisfies *all* of the conclusions.

$$(p \land q) \models \{p, q\}$$

Validities

If $\{\} \vDash \varphi$, then φ is valid.

Examples:

$\{\} \vDash p \vee \neg p$

$\{\} \nvDash p$

$\{\} \nvDash p \wedge \neg p$

The empty set of premises is satisfied by every interpretation. Consequently, if it entails a sentence, that sentence must be true in every interpretation, i.e. it is valid.

Vacuity

If Δ is unsatisfiable, then $\Delta \vDash \varphi$ for ***all*** φ.

Examples:

$\{p, \neg p\} \vDash p$

$\{p, \neg p\} \vDash \neg p$

$\{p, \neg p\} \vDash q$

By definition, an unsatisfiable set of sentences is not satisfied by *any* interpretation. Consequently, it is trivially true that every interpretation that satisfies that set satisfies every sentence.

Unsatisfiable assumptions entail everything!!!

Monotonicity

If $\Gamma \vDash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vDash \varphi$.

Example: $\{p, q\} \vDash p \wedge q$

Therefore $\{p, q, r\} \vDash p \wedge q$

The more you know, the more is entailed.

Ramification

If $\Omega \models \Delta$ and $\Gamma \subseteq \Delta$, then $\Omega \models \Gamma$.

Example: $\{p \wedge q\} \models \{p, q\}$.

Therefore $\{p \wedge q\} \models \{p\}$.

If you can conclude more, you can conclude less.

Third Comparison of Sentences

Constants		Premises	
p	**q**	**p | q**	**~p | ~q**
1	1	1	0
1	0	1	1
0	1	1	1
0	0	0	1

Logical Consistency

A sentence ϕ is *consistent with* a sentence ψ if and only if there is a truth assignment that satisfies both ϕ and ψ.

p is logically consistent with q

$(p \vee q)$ is logically consistent with $(\neg p \vee \neg q)$

$(p \Rightarrow q)$ is logically consistent with $(\neg p \vee q)$

p is *not* consistent with $\neg p$

Is $(p \wedge \neg p)$ logically consistent with $(q \wedge \neg q)$?

Is $(p \wedge \neg p)$ logically consistent with $(p \wedge \neg p)$?

Connections

Propositional Metatheorems

*A metatheorem is a theorem **about** logic.*

Monotonicity Theorem
Ramification Theorem

Equivalence Theorem
Substitution Theorem
Deduction Theorem

Unsatisfiability Theorem
Consistency Theorem

Monotonicity Theorem

If $\Gamma \vDash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vDash \varphi$.

Example: $\{p, q\} \vDash p \wedge q$

Therefore $\{p, q, r\} \vDash p \wedge q$

The more you know, the more is entailed.

Ramification Theorem

If $\Omega \vDash \Delta$ and $\Gamma \subseteq \Delta$, then $\Omega \vDash \Gamma$.

Example: $\{p \wedge q\} \vDash \{p, q\}$

Therefore $\{p \wedge q\} \vDash \{p\}$

If you can conclude more, you can conclude less.

Equivalence Theorem

Theorem: A sentence ϕ and a sentence ψ are *logically equivalent* if and only if the sentence $(\phi \Leftrightarrow \psi)$ is *valid*.

$\neg(p \wedge q)$ is logically equivalent to $(\neg p \vee \neg q)$
if and only if
$(\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q))$ is valid

Upshot: We can determine equivalence of sentences by checking validity of a single sentence.

Upshot: We can demonstrate validity of a biconditional by checking equivalence of the constituents.

Equivalence Theorem

<table>
<tr><th colspan="2">Constants</th><th colspan="2">Premises</th><th>Conclusions</th></tr>
<tr><th>p</th><th>q</th><th>~(p & q)</th><th>~p | ~q</th><th>~(p & q) <=> ~p | ~q</th></tr>
<tr><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Substitution Theorem

Let $\chi_{\varphi \leftarrow \psi}$ stand for a copy of χ where zero or more occurrences of φ have been replaced by ψ.

Example: Let $\chi = (\neg\neg p \lor q)$, then $\chi_{\neg\neg p \leftarrow p} = (p \lor q)$.

Substitution Theorem: If $(\varphi \Leftrightarrow \psi)$ is valid, then the sentence $\chi_{\varphi \leftarrow \psi}$ is logically equivalent to χ.

Example: Since $(p \Leftrightarrow \neg\neg p)$ is valid, we know that the sentence $(\neg\neg p \lor q)$ is logically equivalent to $(p \lor q)$.

Substitution Example

Babbage | Show Instructions

Premises:

```
~~p|q
p|q
```

Truth Table

Constants		Premises	
p	**q**	**~~p | q**	**p | q**
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	0

Deduction Theorem

Theorem: A sentence ϕ *logically entails* a sentence ψ if and only if $(\phi \Rightarrow \psi)$ is *valid*.

More generally, a finite set of sentences $\{\phi_1, \ldots, \phi\}$ logically entails ϕ if and only if the compound sentence $(\phi_1 \wedge \ldots \wedge \phi_n \Rightarrow \phi)$ is valid.

Is $((p \Rightarrow q) \wedge (m \Rightarrow p \vee q) \Rightarrow (m \Rightarrow q))$ valid?
$\{(p \Rightarrow q), (m \Rightarrow p \vee q)\} \vDash (m \Rightarrow q)$?

Upshot: We can determine logical entailment between sentences by checking validity of a single sentence. *And vice versa*.

Deduction Theorem

$\{(m \Rightarrow p \vee q), (p \Rightarrow q)\} \vDash (m \Rightarrow q)$?

Is $((m \Rightarrow p \vee q) \wedge (p \Rightarrow q) \Rightarrow (m \Rightarrow q))$ valid?

Constants			Premises			Conclusions
m	**p**	**q**	**m => p | q**	**p => q**	**m => q**	**(m => p | q) & (p => q) => (m => q)**
1	1	1	1	1	1	1
1	1	0	1	0	0	1
1	0	1	1	1	1	1
1	0	0	0	1	0	1
0	1	1	1	1	1	1
0	1	0	1	0	1	1
0	0	1	1	1	1	1
0	0	0	1	1	1	1

Unsatisfiability Theorem

Theorem: $\Delta \models \varphi$ if and only if $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Proof: Suppose that $\Delta \models \varphi$. If an interpretation satisfies Δ, then it must also satisfy φ. But then it cannot satisfy $\neg\varphi$. Therefore, $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Suppose that $\Delta \cup \{\neg\varphi\}$ is unsatisfiable. Then every interpretation that satisfies Δ must *fail* to satisfy $\neg\varphi$, i.e. it must satisfy φ. Therefore, $\Delta \models \varphi$.

Upshot: We can determine logical entailment between sentences by checking unsatisfiability of a set of sentences.

Translation: Assume false and show contradiction.

Consistency Theorem

Theorem: A sentence ϕ is logically *consistent* with a sentence ψ if and only if the sentence $(\phi \wedge \psi)$ is *satisfiable*. More generally, a sentence ϕ is logically consistent with a finite set of sentences $\{\phi_1, \ldots, \phi_n\}$ if and only if the compound sentence $(\phi_1 \wedge \ldots \wedge \phi_n \wedge \phi)$ is satisfiable.

Is $(p \vee q)$ consistent with $(\neg p \vee \neg q)$?
Is $((p \vee q) \wedge (\neg p \vee \neg q))$ satisfiable?

Upshot: We can determine consistency of sentences by checking satisfiability of a single sentence.

Metareasoning

CS 157 Quiz Question #1

Is the sentence $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ valid, contingent, or unsatisfiable?

CS 157 Quiz Question #1

Is the sentence $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ valid, contingent, or unsatisfiable?

p	q	$(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$
1	1	1
1	0	0
0	1	1
0	0	1

CS 157 Quiz Question #1

Is the sentence $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ valid, contingent, or unsatisfiable?

$(p \Rightarrow q)$ is sometimes true and sometimes false.

$(p \Rightarrow (q \Rightarrow p))$ is always true, i.e. it is a valid sentence.

$(p \Rightarrow q)^i = (p \Rightarrow (q \Rightarrow p))^i$ for some i.

$(p \Rightarrow q)^i \neq (p \Rightarrow (q \Rightarrow p))^i$ for some i.

$(p \Rightarrow q) \Leftrightarrow (p \Rightarrow (q \Rightarrow p))$ is contingent.

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

p	q	$((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$
1	1	1
1	0	1
0	1	1
0	0	1

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

$$((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$$

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

$$(p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q)$$

$$(p \Leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

Substitution Theorem
and
$(p \Rightarrow \neg q) \Leftrightarrow (\neg p \mid \neg q)$

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

$$((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$$

$$(p \Leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \land (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

Substitution Theorem
and
$(p \Leftrightarrow \neg q) \Leftrightarrow (p \Rightarrow \neg q) \land (\neg q \Rightarrow p)$

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

$$((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$$

$$(p \Leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \models (p \Rightarrow \neg q)$$

Deduction Theorem

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

$$((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$$

$$(p \Leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \quad \models \quad (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q), (\neg q \Rightarrow p)\} \quad \models \quad (p \Rightarrow \neg q)$$

Definition of Entailment
Definition of Conjunction

CS 157 Quiz Question #2

Is the sentence $((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$ valid, contingent, or unsatisfiable?

$$((p \Leftrightarrow \neg q) \Rightarrow (\neg p \mid \neg q))$$

$$(p \Leftrightarrow \neg q) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \Rightarrow (p \Rightarrow \neg q)$$

$$(p \Rightarrow \neg q) \wedge (\neg q \Rightarrow p) \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q), (\neg q \Rightarrow p)\} \models (p \Rightarrow \neg q)$$

$$\{(p \Rightarrow \neg q)\} \models (p \Rightarrow \neg q)$$

Monotonicity Theorem

CS 157 Quiz Question #3a

Let Γ and Δ be arbitrary sets of sentences.

Let φ be an arbitrary sentence.

If $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cup \Delta \models \varphi$?

CS 157 Quiz Question #3a

Let Γ and Δ be arbitrary sets of sentences.
Let φ be an arbitrary sentence.
Is $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cup \Delta \models \varphi$?

Let Γ be $\{p\}$ and Δ be $\{q\}$ and φ be $(p \vee q)$.

Obviously $\{p\} \models p \vee q$
Obviously $\{q\} \models p \vee q$

But $\{p\} \cup \{q\} = \{p, q\}$ and $\{p, q\} \models p \vee q$

Does this work for all Γ and Δ and φ?
Yes, by Monotonicity Theorem.

CS 157 Quiz Question #3b

Let Γ and Δ be arbitrary sets of sentences.

Let φ be an arbitrary sentence.

Is $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cap \Delta \models \varphi$?

CS 157 Quiz Question #3b

Let Γ and Δ be arbitrary sets of sentences.
Let φ be an arbitrary sentence.
Is $\Gamma \models \varphi$ and $\Delta \models \varphi$, does $\Gamma \cap \Delta \models \varphi$?

Let Γ be $\{p\}$ and Δ be $\{q\}$ and φ be $(p \vee q)$.

Obviously $\{p\} \models p \vee q$
Obviously $\{q\} \models p \vee q$

But $\{p\} \cap \{q\} = \{\}$ and $\{\} \not\models p \vee q$.

Answer to our question: No.

Tools

Course Website

http://logica.stanford.edu

Babbage

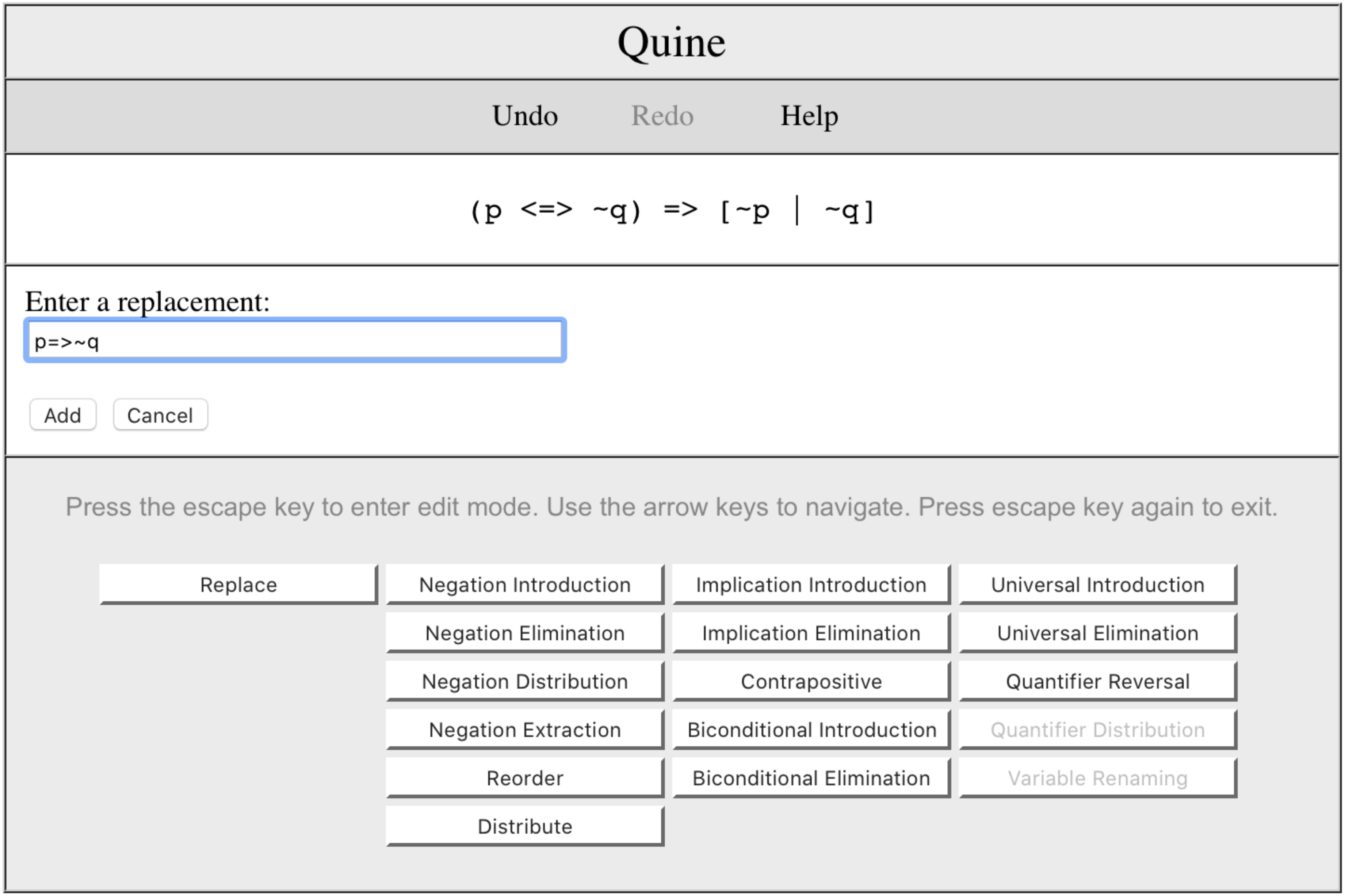
Show Instructions

Premises:

```
((p <=> ~q) => (~p | ~q))
```

Truth Table

<table>
<tr><th colspan="2">Constants</th><th>Premises</th></tr>
<tr><th>p</th><th>q</th><th>(p <=> ~q) => ~p | ~q</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
</table>


Quine
Undo
Redo
Help
(p <=> ~q) => [~p | ~q]
Enter a replacement:
p=>~q
Add
Cancel
Press the escape key to enter edit mode. Use the arrow keys to navigate. Press escape key again to exit.
Replace
Negation Introduction
Implication Introduction
Universal Introduction
Negation Elimination
Implication Elimination
Universal Elimination
Negation Distribution
Contrapositive
Quantifier Reversal
Negation Extraction
Biconditional Introduction
Quantifier Distribution
Reorder
Biconditional Elimination
Variable Renaming
Distribute