Introduction to Logic
Propositional Logic

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Multiple Logics

→ Propositional Logic (logical operators)

*If **raining** and **cold**, then **wet**.*

Relational Logic (variables and quantifiers)

*If **abby** likes **x**, then **bess** likes **x**.*

Functional Logic (functional terms)

*{**a**, **b**} is a subset of {**a**, **b**, **c**}.*

Example

If Mary loves Pat, then Mary loves Quincy.

If it is Monday, then Mary loves Pat or Quincy.

If it is Monday, does Mary love Quincy?

If it is Monday, does Mary love Pat?

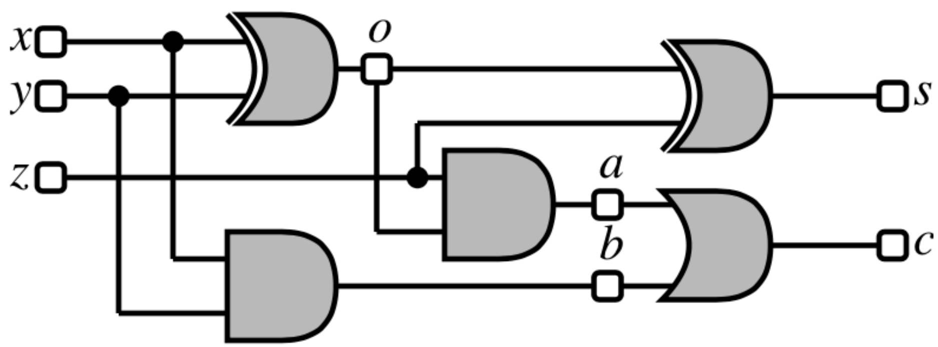
Example

Victor has been murdered, and Art, Bob, and Carl are suspects. Art says he did not do it. He says that Bob was the victim's friend but that Carl hated the victim. Bob says he was out of town the day of the murder, and besides he didn't even know the guy. Carl says he is innocent and he saw Art and Bob with the victim just before the murder. You can assume that everyone is telling the truth - except for the murderer.

Whodunnit? (Answer: Bob)

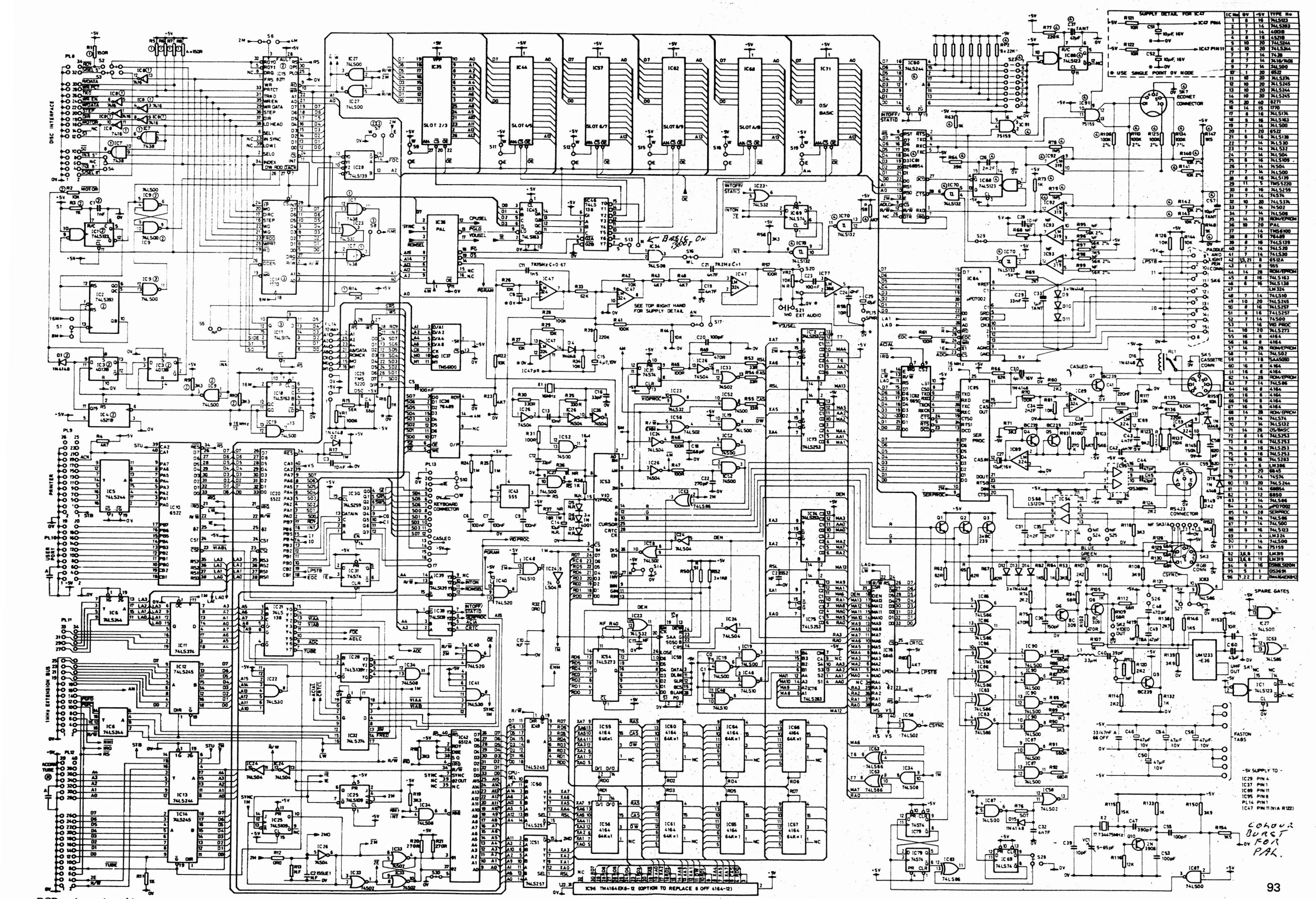
http://intrologic.stanford.edu/extras/whodunnit.html

Digital Circuits



http://intrologic.stanford.edu/extras/circuits.html

Digital Circuits Example



PCB circuit diagram

Programme

Basics

- syntactically legal sentences
- meaning of syntactically legal sentences

Evaluation

- Given truth values for simple sentences,
- find truth values of complex sentences.

Satisfaction

- Given truth values for complex sentences,
- find truth values of simple sentences.

Examples

- Natural Language
- Digital Circuits

Programme

Basics

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Examples

- Natural Language
- Digital Circuits

Syntax

Propositional Sentences

Proposition Constants

express simple facts about the world

In English - *It is raining*.

In our language - *raining*

Compound Sentences

express relationships between sentences

In English - *It is raining* ***or*** *it is snowing*.

In our language - *raining* ∨ *snowing*

Proposition Constants

By convention (in this course), proposition constants are written as strings of letters, digits, and underscores ("_") beginning with a lower case letter.

Examples:

raining
rAiNiNg
raining_or_snowing

Non-Examples:

Raining
324567
raining-or-snowing

Compound Sentences (part I)

Negations:

$$(\neg raining)$$

The argument of a negation is called the *target*.

Conjunctions:

$$(raining \wedge snowing)$$

The arguments of a conjunction are called *conjuncts*.

Disjunctions:

$$(raining \vee snowing)$$

The arguments of a disjunction are called *disjuncts*.

Compound Sentences (part II)

Implications:

$$(raining \Rightarrow cloudy)$$

The left argument of an implication is the *antecedent*.
The right argument is the *consequent*.

Biconditionals:

$$(cloudy \Leftrightarrow raining)$$

Nested Compound Sentences

$(\neg raining)$

$(raining \wedge snowing)$

$(raining \vee snowing)$

$(raining \Rightarrow cloudy)$

$(cloudy \Leftrightarrow raining)$

$\neg(raining \wedge snowing)$

$((raining \wedge snowing) \Rightarrow cloudy)$

$(cloudy \Rightarrow (raining \wedge snowing))$

$((cloudy \wedge wet) \Leftrightarrow (raining \vee snowing))$

$(\neg raining \Rightarrow (cloudy \Rightarrow snowing))$

Parentheses Removal

Dropping Parentheses is good:

$$(p \wedge q) \rightarrow p \wedge q$$

But it can lead to ambiguities:

$$((p \vee q) \wedge r) \rightarrow p \vee q \wedge r$$

$$(p \vee (q \wedge r)) \rightarrow p \vee q \wedge r$$

Precedence

Parentheses can be dropped when the structure of an expression can be determined by precedence.

$$\neg$$
$$\wedge$$
$$\vee$$
$$\Rightarrow$$
$$\Leftrightarrow$$

An operand surrounded by operators associates with operator of higher precedence.

$$\neg p \vee q \rightarrow ((\neg p) \vee q)$$
$$p \vee q \wedge r \rightarrow (p \vee (q \wedge r))$$
$$p \wedge q \Rightarrow r \rightarrow ((p \wedge q) \Rightarrow r)$$
$$p \Rightarrow q \Leftrightarrow r \rightarrow ((p \Rightarrow q) \Leftrightarrow r)$$

Precedence (continued)

If surrounded by occurrences of $\wedge$ or $\vee$, the operand associates with the operator to the left.

$$p \wedge q \wedge r \rightarrow ((p \wedge q) \wedge r)$$
$$p \vee q \vee r \rightarrow ((p \vee q) \vee r)$$

If surrounded by two occurrences of $\Rightarrow$ or $\Leftrightarrow$, the operand associates with the operator to the right.

$$p \Rightarrow q \Rightarrow r \rightarrow (p \Rightarrow (q \Rightarrow r))$$
$$p \Leftrightarrow q \Leftrightarrow r \rightarrow (p \Leftrightarrow (q \Leftrightarrow r))$$

Useful Definitions

A *propositional vocabulary* is a set of *proposition constants*.

Given a propositional vocabulary, a *propositional sentence* is either

(1) a proposition constant or

(2) a compound sentence.

A *propositional language* is the set of *all* propositional sentences that can be formed from a propositional vocabulary.

Semantics

Propositional Interpretation

A *propositional interpretation* is an association between the propositional constants in a propositional language and the values T or F.

$$p \xrightarrow{i} T \qquad p^i = T$$

$$q \xrightarrow{i} F \qquad q^i = F$$

$$r \xrightarrow{i} T \qquad r^i = T$$

We sometimes write 1 and 0 in place of T and F.

$$p^i = 1$$

$$q^i = 0$$

$$r^i = 1$$

Sentential Interpretation

A *sentential interpretation* is an association between the sentences in a propositional language and truth values.

$p^i = \mathrm{T}$

$q^i = \mathrm{F}$

$r^i = \mathrm{T}$

$(p \vee q)^i = \mathrm{T}$

$(\neg q \vee r)^i = \mathrm{T}$

$((p \vee q) \wedge (\neg q \vee r))^i = \mathrm{T}$

NB: Each distinct propositional interpretation gives rise to a unique sentential interpretation due to operator semantics.

Semantics of Negations

A *negation* is true if and only if the target is false.

ϕ	$\neg\phi$
T	F
F	T

For example, if the interpretation of p is F, then the interpretation of $\neg p$ is T.

For example, if the interpretation of $(p \wedge q)$ is T, then the interpretation of $\neg(p \wedge q)$ is F.

Semantics of Conjunctions

A *conjunction* is true if and only if both conjuncts are true.

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

For example, if the interpretation of p is true and q is true, then $(p \wedge q)$ is true.

Semantics of Disjunctions

A *disjunction* is true if and only if at least one of the disjuncts is true.

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

The type of disjunction here is called *inclusive or*. This contrasts with *exclusive or*, which says that a disjunction is true if and only if an *odd number* of disjuncts are true.

Semantics of Biconditionals

A *biconditional* is true if and only if the truth values of its two constituents are the same.

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Semantics of Implications

An *implication* is true if and only if the antecedent is false *or* the consequent is true.

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

The semantics of implication here is called *material implication*.

What Choice Do We Have?

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

ϕ	ψ	ψ
T	T	T
T	F	F
F	T	T
F	F	F

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Implications and Biconditionals

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	ψ	$\psi \Rightarrow \varphi$
T	T	T
T	F	T
F	T	F
F	F	T

$\varphi \Leftrightarrow \psi$ is true if and only if $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$ are true.

ϕ	ψ	$\phi \Leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Counterfactuals

An *implication* is true if and only if the antecedent is false or the consequent is true.

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

A *counterfactual* is an implication in which the antecedent is false.

Counterfactuals are Weird

A *counterfactual* is an implication in which the antecedent is false.

NB: Counterfactuals are always *true!* The truth value of the consequent does not matter.

Examples:

$$\textit{It is raining} \Rightarrow \textit{I am a billionaire}$$
$$\textit{It is raining} \Rightarrow \textit{I am a pauper}$$
$$\textit{Shakespeare is alive} \Rightarrow \textit{Shakespeare is dead}$$
$$2+2=5 \Rightarrow 2+2=7$$

Evaluation

Evaluation

Interpretation i:

$$p^i = \mathrm{T}$$
$$q^i = \mathrm{T}$$
$$r^i = \mathrm{F}$$

Compound Sentence

$$(p \lor q) \land (\neg q \lor r)$$
$$(\mathrm{T} \lor \mathrm{T}) \land (\neg \mathrm{T} \lor \mathrm{F})$$
$$(\mathrm{T} \lor \mathrm{T}) \land (\ \mathrm{F} \lor \mathrm{F})$$
$$\mathrm{T} \quad \land \quad \mathrm{F}$$
$$\mathrm{F}$$

Evaluation

Interpretation i:

$$p^i = \mathrm{T}$$
$$q^i = \mathrm{F}$$
$$r^i = \mathrm{T}$$

Compound Sentence

$$(p \vee q) \wedge (\neg q \vee r)$$
$$(\mathrm{T} \vee \mathrm{F}) \wedge (\neg \mathrm{F} \vee \mathrm{T})$$
$$(\mathrm{T} \vee \mathrm{F}) \wedge (\mathrm{T} \vee \mathrm{T})$$
$$\mathrm{T} \quad \wedge \quad \mathrm{T}$$
$$\mathrm{T}$$

Evaluation

Interpretation i:

$$p^i = \mathrm{T}$$
$$q^i = \mathrm{F}$$
$$r^i = \mathrm{T}$$

Compound Sentence

$$(p \wedge q) \vee (\neg q \wedge r)$$
$$(\mathrm{T} \wedge \mathrm{F}) \vee (\neg \mathrm{F} \wedge \mathrm{T})$$
$$(\mathrm{T} \wedge \mathrm{F}) \vee (\mathrm{T} \wedge \mathrm{T})$$
$$\mathrm{F} \quad \vee \quad \mathrm{T}$$
$$\mathrm{T}$$

Satisfaction

Evaluation versus Satisfaction

Evaluation:

$$\begin{array}{lcl} p^i & = & \mathrm{T} \\ q^i & = & \mathrm{F} \end{array} \longrightarrow \begin{array}{lcl} (p \vee q)^i & = & \mathrm{T} \\ (\neg q)^i & = & \mathrm{T} \end{array}$$

Satisfaction:

$$\begin{array}{lcl} (p \vee q)^i & = & \mathrm{T} \\ (\neg q)^i & = & \mathrm{T} \end{array} \longrightarrow \begin{array}{lcl} p^i & = & \mathrm{T} \\ q^i & = & \mathrm{F} \end{array}$$

Multiple Interpretations

Logic does not prescribe which interpretation is "correct". In the absence of additional information, one interpretation is as good as another.

Interpretation i			Interpretation j		
p^i	=	T	p^j	=	F
q^i	=	F	q^j	=	F
r^i	=	T	r^j	=	T

Examples:

- Different days of the week
- Different locations
- Beliefs of different people

Truth Tables

A *truth table* is a table of all possible interpretations for the propositional constants in a language.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

One column per constant.

One row per interpretation.

For a language with n constants, there are 2^n interpretations.

Truth Table Method

Method to find all propositional interpretations that satisfy a given set of sentences:

(1) Form a truth table for the propositional constants.

(2) For each sentence in the set and each row in the truth table, check whether the row satisfies the sentence. If not, cross out the row.

(3) Any row remaining satisfies all sentences in the set. (Note that there might be more than one.)

Are these sentences satisfiable?

$$q \Rightarrow r$$

$$p \Rightarrow q \wedge r$$

$$\neg r$$

Satisfaction Example

$$p \Rightarrow q \wedge r$$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Satisfaction Example

$q \Rightarrow r$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

$p \Rightarrow q \wedge r$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Satisfaction Example

$q \Rightarrow r$

p	q	r
T	T	T
T	**T**	**F**
T	F	T
T	F	F
F	T	T
F	**T**	**F**
F	F	T
F	F	F

$p \Rightarrow q \wedge r$

p	q	r
T	T	T
T	**T**	**F**
T	**F**	**T**
T	**F**	**F**
F	T	T
F	T	F
F	F	T
F	F	F

$\neg r$

p	q	r
T	**T**	**T**
T	T	F
T	**F**	**T**
T	F	F
F	**T**	**T**
F	T	F
F	**F**	**T**
F	F	F

Satisfaction Example

$$\{q \Rightarrow r,\ p \Rightarrow q \wedge r,\ \neg r\}$$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Logica

Course Website

http://logica.stanford.edu

Not Secure — logica.stanford.edu

Logica

Quine Equivalence Editor	Babbage Truth Table Generator	Hilbert Hilbert-style Proof Editor
Stickel Clausal Form Converter	Boole Truth Table Comparator	Fitch Fitch-style Proof Editor
Wegman Unifier	Clarke Logic Grid Editor	Robinson Resolution Proof Editor
	Russell Constraint Satisfier	
	Herbrand Sentence Analyzer	

Babbage

Show Instructions

Premises:

```
q=>r
p => q&r
~r
```

Truth Table

Constants			Premises		
q	**r**	**p**	**q => r**	**p => q & r**	**~r**
1	1	1	1	1	0
1	1	0	1	1	0
1	0	1	0	0	1
1	0	0	0	1	1
0	1	1	1	0	0
0	1	0	1	1	0
0	0	1	1	0	1
0	0	0	1	1	1

Interleaved Generation and Checking

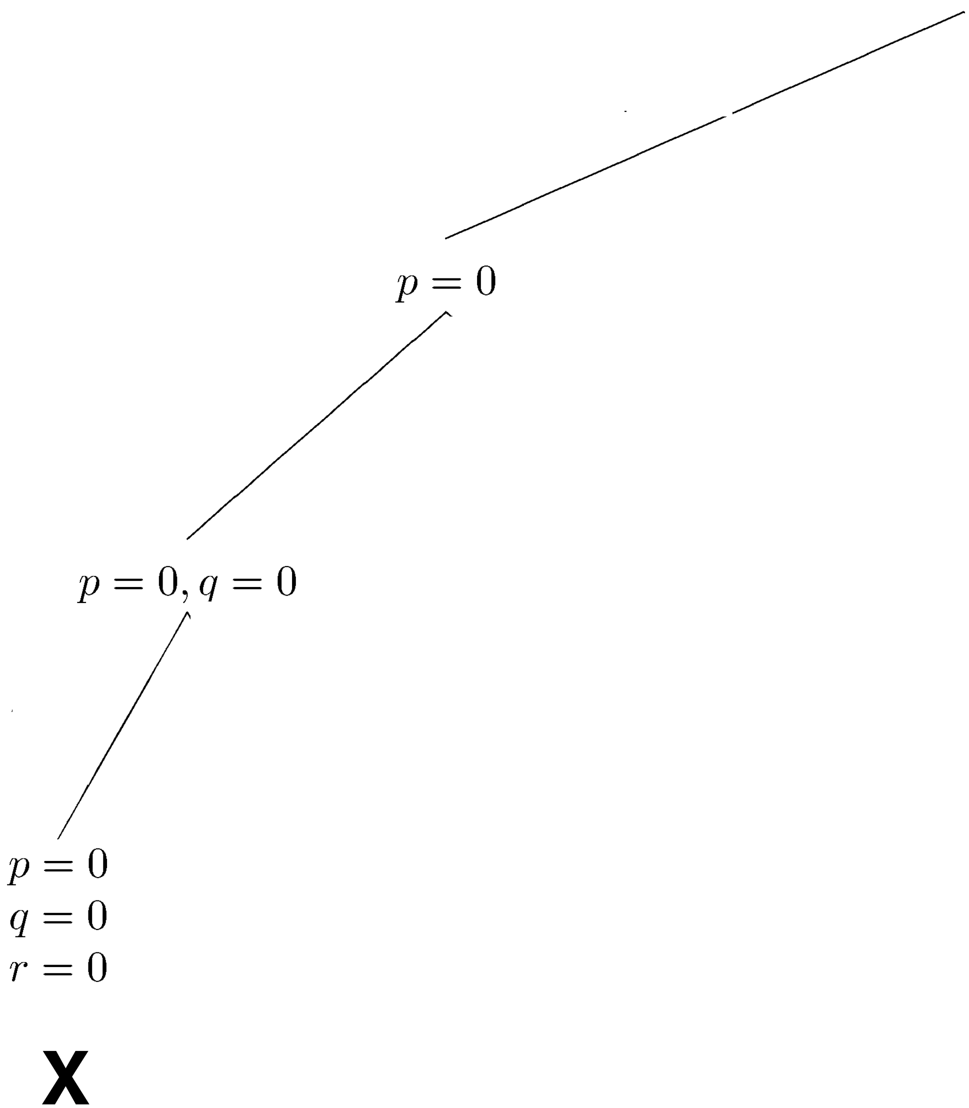
Generation then Evaluation

$$\{p \vee q,\; p \vee \neg q,\; \neg p \vee q,\; \neg p \vee \neg q \vee \neg r,\; \neg p \vee r\}$$

p	q	r	$p \vee q$	$p \vee \neg q$	$\neg p \vee q$	$\neg p \vee \neg q \vee \neg r$	$\neg p \vee r$	Δ
0	0	0	0	1	1	1	0	0
0	0	1	0	1	1	1	1	0
0	1	0	1	0	1	1	1	0
0	1	1	1	0	1	1	1	0
1	0	0	1	1	0	1	0	0
1	0	1	1	1	0	1	1	0
1	1	0	1	1	1	1	0	0
1	1	1	1	1	1	0	1	0

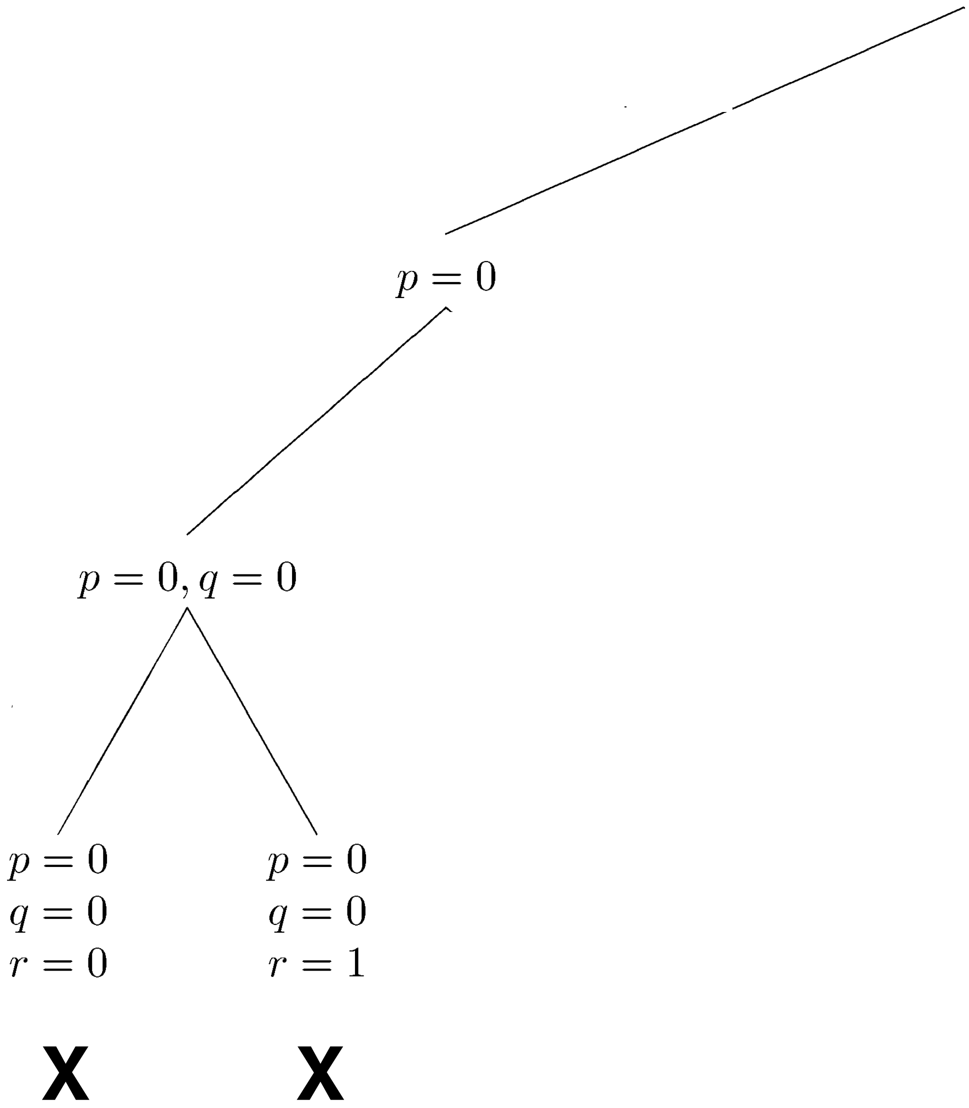
Interleaved Generation and Evaluation

$$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$$



Interleaved Generation and Evaluation

$$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$$



Interleaved Generation and Evaluation

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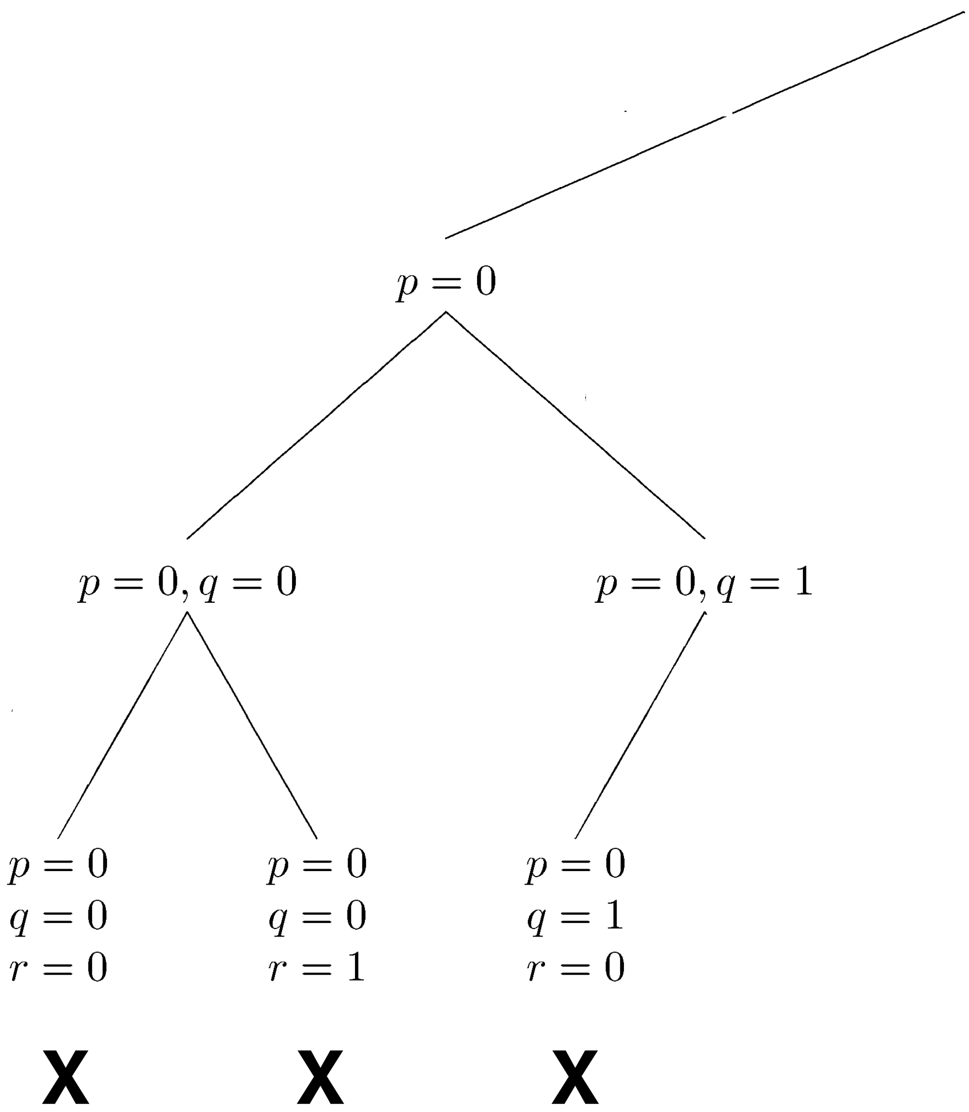

Interleaved Generation and Evaluation

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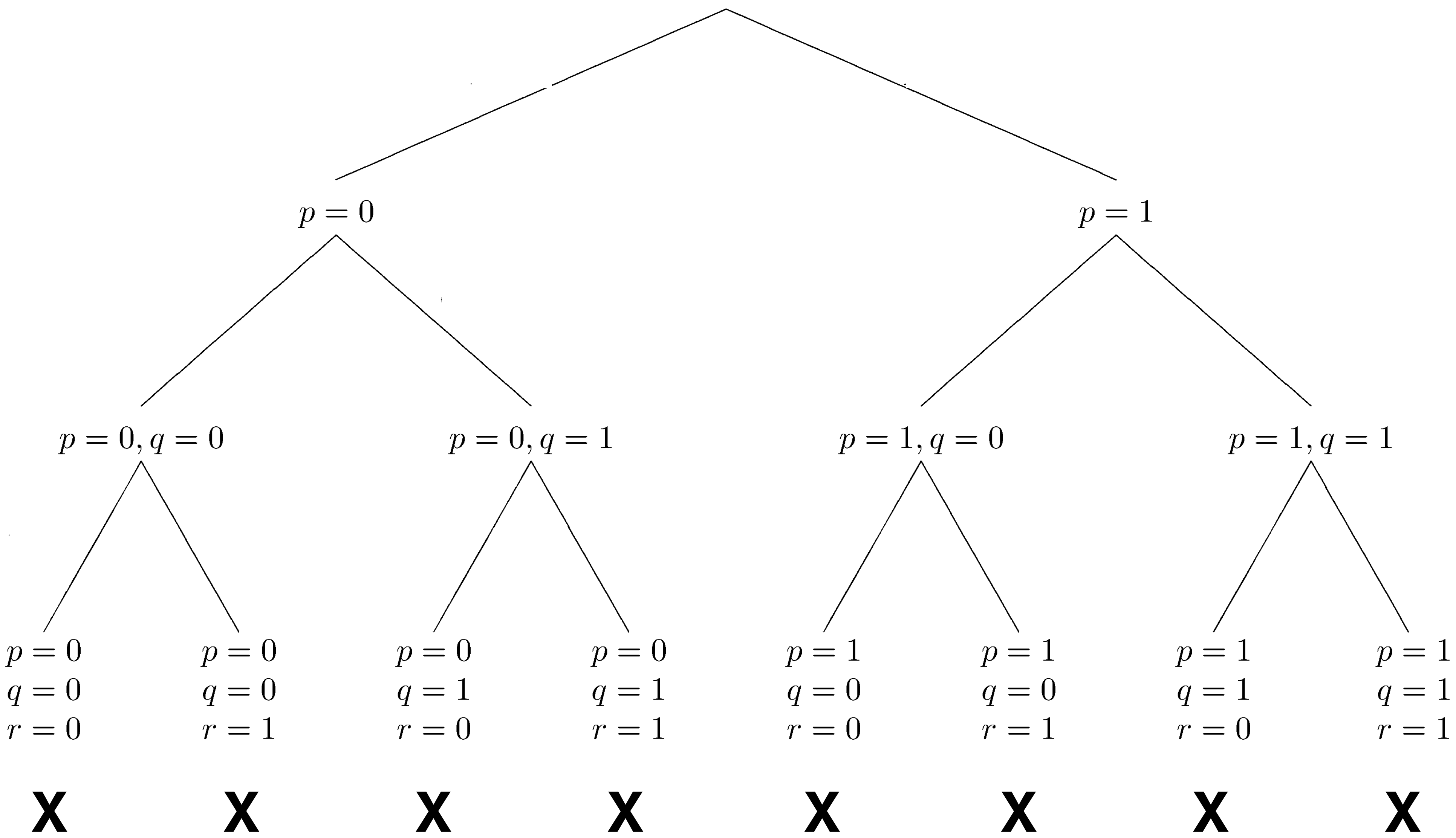
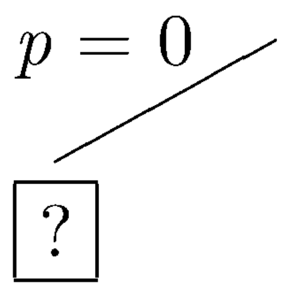


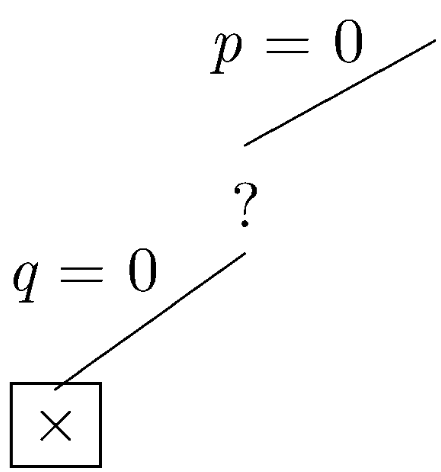
Intermediate State Checking

$$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$$



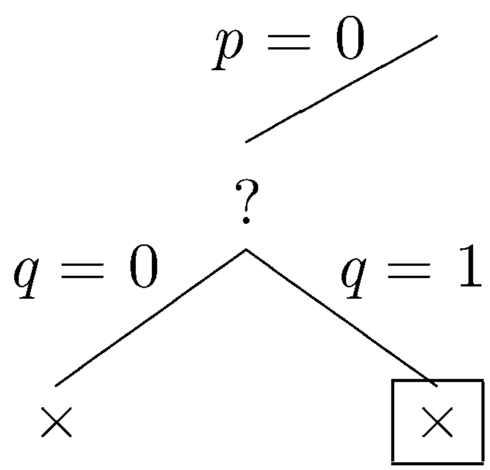
Intermediate State Checking

$$\{p \lor q,\ p \lor \neg q,\ \neg p \lor q,\ \neg p \lor \neg q \lor \neg r,\ \neg p \lor r\}$$



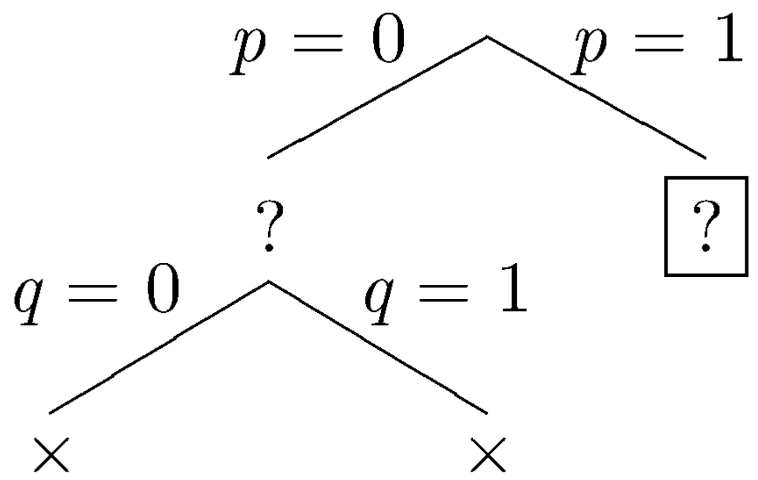
Intermediate State Checking

$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$



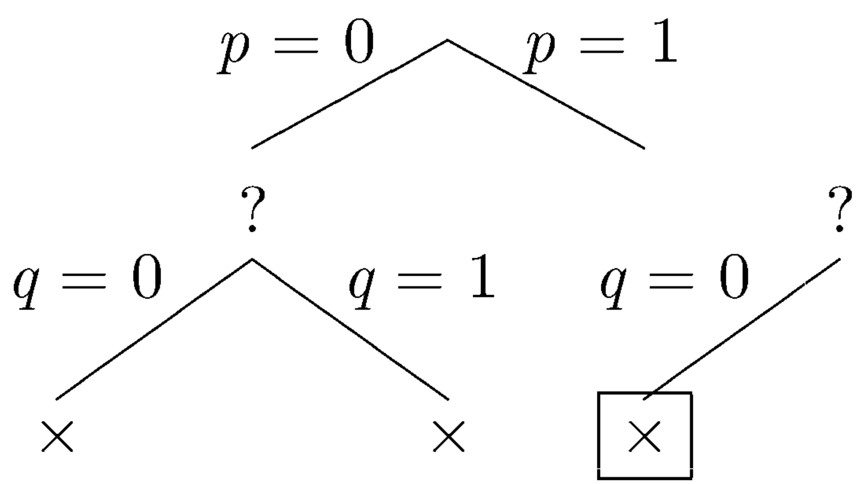
Intermediate State Checking

$\{p \lor q,\ p \lor \neg q,\ \neg p \lor q,\ \neg p \lor \neg q \lor \neg r,\ \neg p \lor r\}$



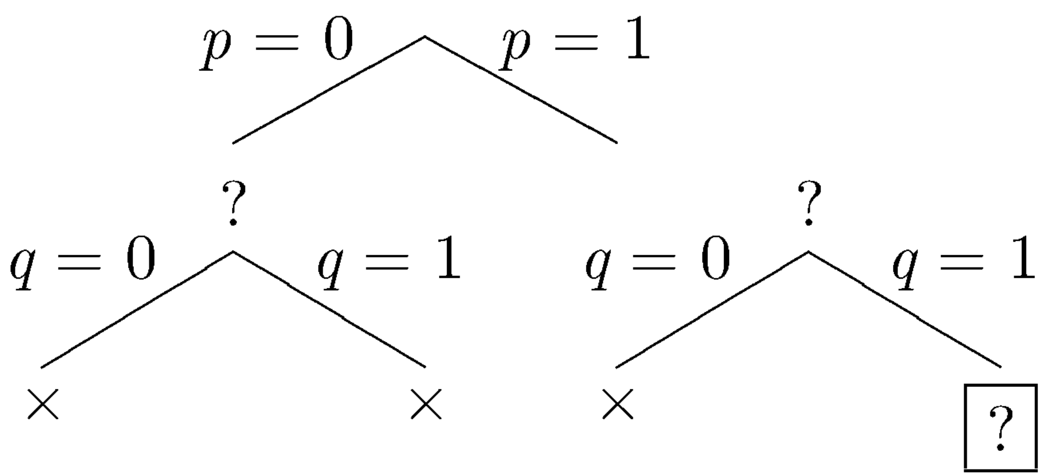
Intermediate State Checking

$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$



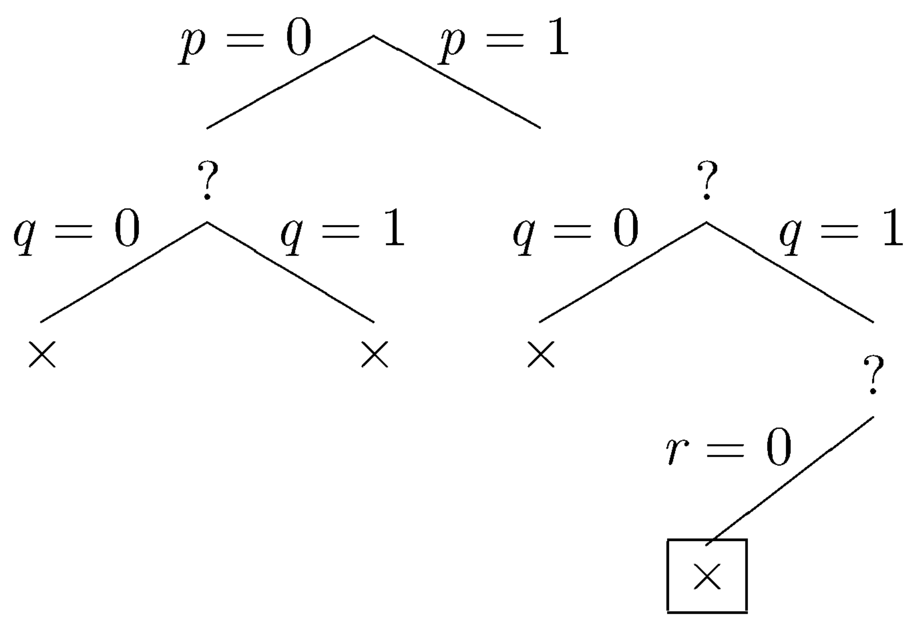
Intermediate State Checking

$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$



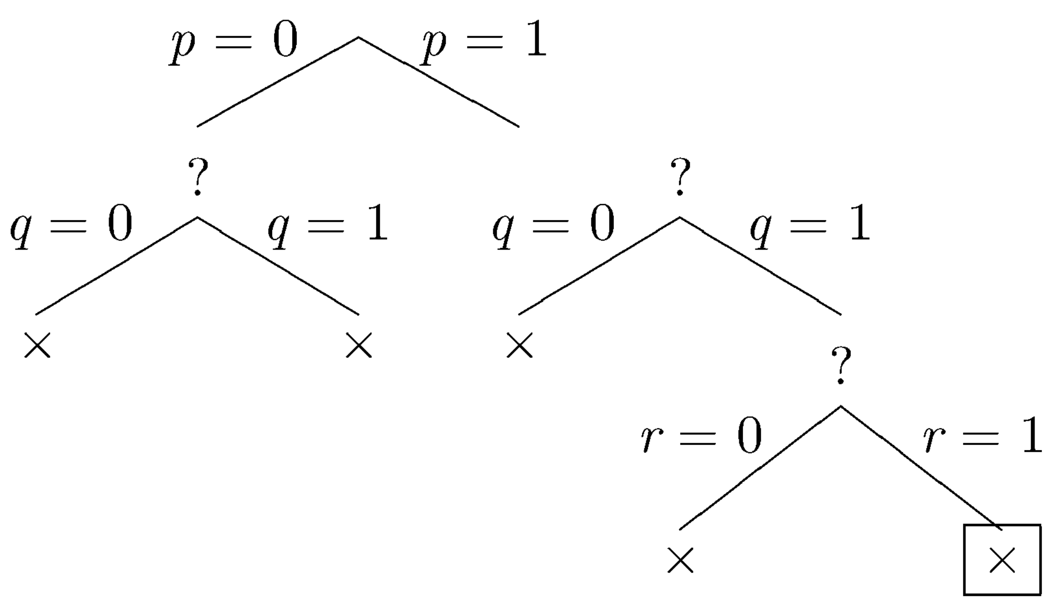
Intermediate Checking

$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$



Intermediate Checking

$\{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}$



Simplification and Unit Propagation

Simplification

Constraints

$p \vee q$

$p \vee {\sim}q$

${\sim}p \vee q$

${\sim}p \vee {\sim}q \vee {\sim}r$

${\sim}p \vee r$

Simplification

Given $p = 1$

Original	**Simplified**
$p \lor q$	-
$p \lor \sim q$	-
$\sim p \lor q$	q
$\sim p \lor \sim q \lor \sim r$	$\sim q \lor \sim r$
$\sim p \lor r$	r

Unit Propagation

Given $p = 1, q = 1$

Original	**Simplified**
$p \lor q$	-
$p \lor \sim q$	-
$\sim p \lor q$	-
$\sim p \lor \sim q \lor \sim r$	$\sim r$
$\sim p \lor r$	r

Simplification

Given $p = 1, q = 1, r = 1$

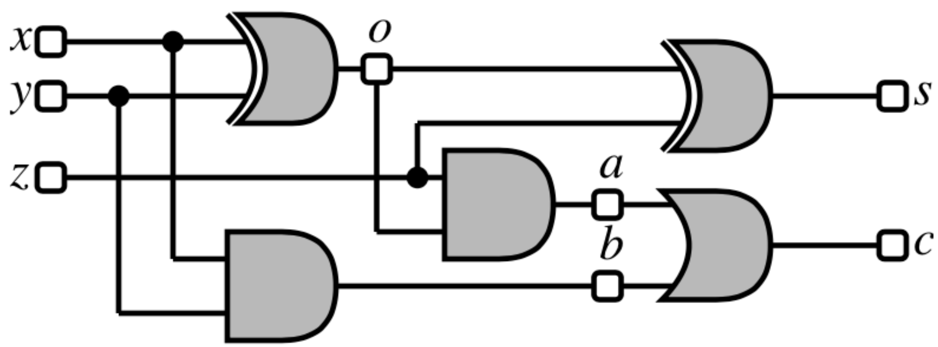
Original	**Simplified**
$p \vee q$	-
$p \vee {\sim}q$	-
${\sim}p \vee q$	-
${\sim}p \vee {\sim}q \vee {\sim}r$	X
${\sim}p \vee r$	-

More on Computing Satisfaction

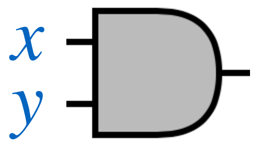
http://intrologic.stanford.edu/extras/satisfiability.html

Digital Circuits

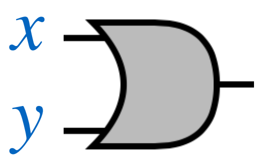
Digital Circuits



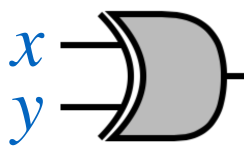
Gates



$x \wedge y$



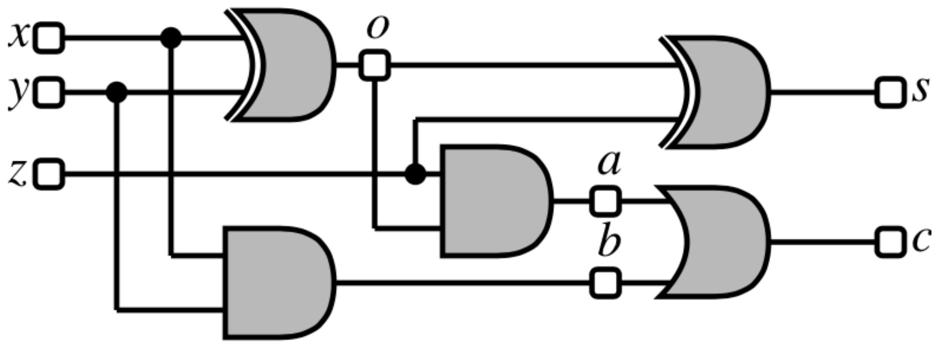
$x \vee y$



$(x \wedge \neg y) \vee (\neg x \wedge y)$

$\neg(x \Leftrightarrow y)$

Example



o: $(x \wedge \neg y) \vee (\neg x \wedge y)$

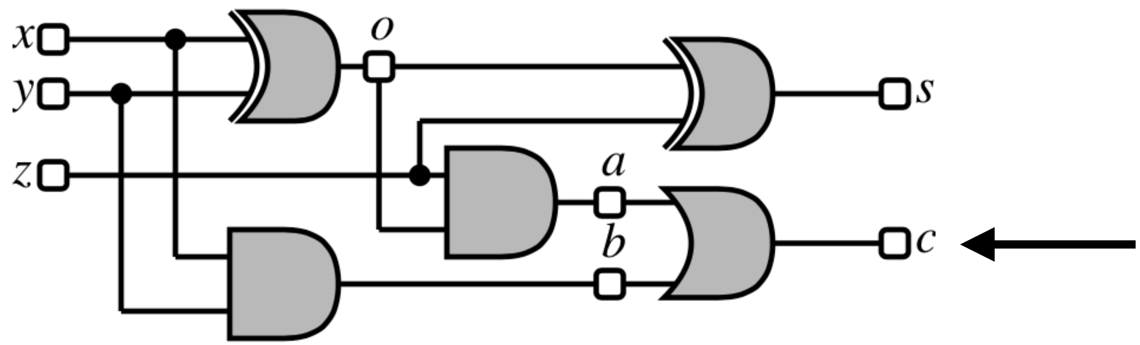
b: $x \wedge y$

a: $z \wedge ((x \wedge \neg y) \vee (\neg x \wedge y))$

c: $(z \wedge ((x \wedge \neg y) \vee (\neg x \wedge y))) \vee (x \wedge y)$

Evaluation Example

$x^i = 1$
$y^i = 0$
$z^i = 1$



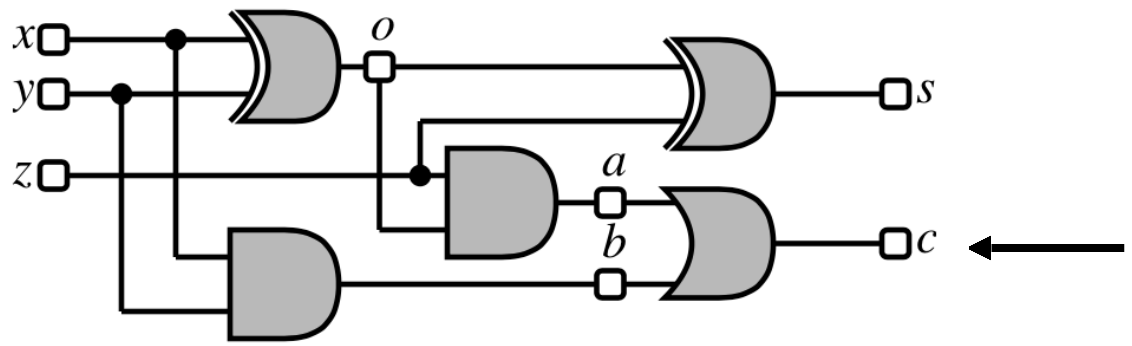
$$[(z \;\wedge\; ((x \wedge \neg y) \vee (\neg x \wedge y))) \;\vee\; (x \wedge y)]^{\,i} = ?$$

Example

$x^i = 1$

$y^i = 0$

$z^i = 1$



$$(z \wedge ((x \wedge \neg y) \vee (\neg x \wedge y))) \vee (x \wedge y)$$

$$(1 \wedge ((1 \wedge \neg 0) \vee (\neg 1 \wedge 0))) \vee (1 \wedge 0)$$

$$(1 \wedge ((1 \wedge 1) \vee (0 \wedge 1))) \vee (1 \wedge 0)$$

$$(1 \wedge (1 \vee 0)) \vee 0$$

$$(1 \wedge 1) \vee 0$$

$$1 \vee 0$$

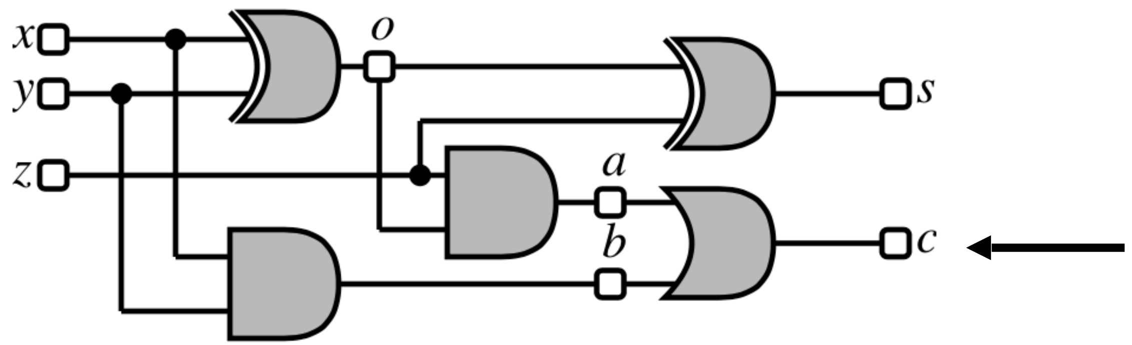
$$1$$

Satisfaction Example

$x^i = ?$

$y^i = ?$

$z^i = ?$

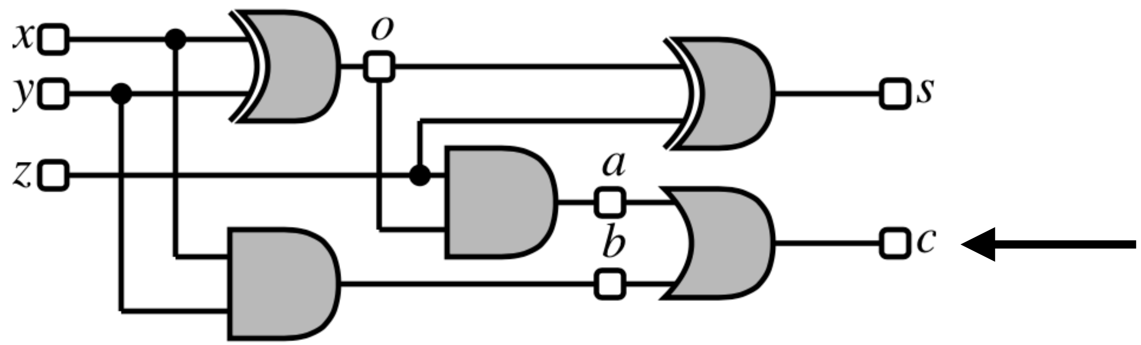


$$((z \wedge ((x \wedge \neg y) \vee (\neg x \wedge y))) \vee (x \wedge y))^i = 1$$

Constants			Premises
z	**x**	**y**	**z & (x & ~y | ~x & y) | x & y**
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

Diagnosis Example

$x^i = 1$
$y^i = 0$
$z^i = 1$



Observation:

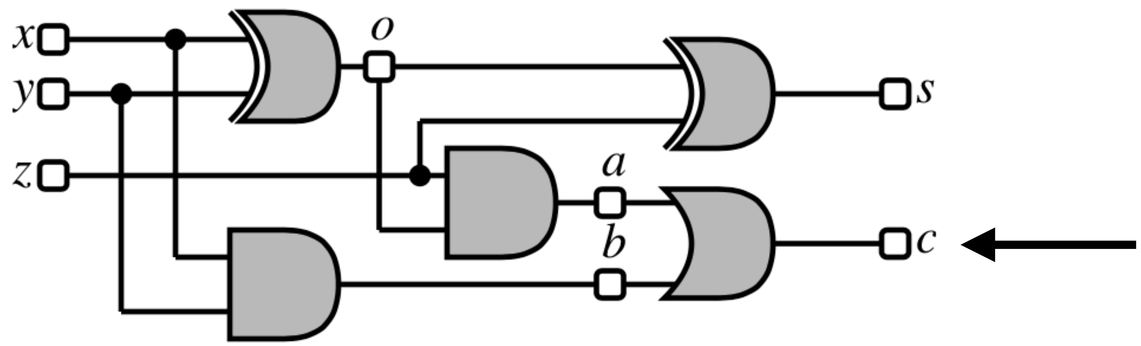
$$[(z \wedge ((x \wedge \neg y) \vee (\neg x \wedge y))) \vee (x \wedge y)]^i = 0$$

Problem:

Which gate is malfunctioning?

Testing Example

$x^i = ?$
$y^i = ?$
$z^i = ?$



$$[(z \ \wedge \ ((x \wedge \neg y) \vee (\neg x \wedge y))) \ \vee \ (x \wedge y)]^{\,i} = ?$$

Problem:

How many inputs must one try to determine whether there is a faulty gate? Answer: fewer than 8.

Digital Circuits Extras

Evaluation, Satisfaction, Diagnosis, Testing

http://intrologic.stanford.edu/extras/circuits.html

Graphical design and simulation

https://logic.ly/demo/

The Big Game

The Big Game

Stanford people always tell the truth, and Berkeley people always lie. Unfortunately, by looking at a person, you cannot tell whether he is from Stanford or Berkeley.

You come to a fork in the road and want to get to the football stadium down one fork. However, you do not know which to take. There is a person standing there. What single question can you ask him to help you decide which fork to take?

Basic Idea

left	*su*	*Question*	*Response*
T	T		
T	F		
F	T		
F	F		

Desired Response

left	*su*	*Question*	*Response*
T	T		"T"
T	F		"T"
F	T		"F"
F	F		"F"

Desired Response

left	*su*	*Question*	*Response*
T	T	T	"T"
T	F	F	"T"
F	T	F	"F"
F	F	T	"F"

The Big Game (solved)

Question: *The left road is the way to the stadium if and only if you are from Stanford. Is that correct?*

$$left \Leftrightarrow su$$

Let's Be Careful Out There