Introduction to Logic
Equality

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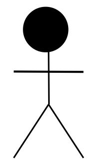
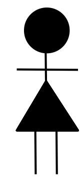
Isomorphic Representation

People

michael $\leftrightarrow$ 

maureen $\leftrightarrow$ 

Arithmetic

$0 \leftrightarrow 0$

$s(0) \leftrightarrow 1$

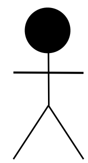
$s(s(0)) \leftrightarrow 2$

$s(s(s(0))) \leftrightarrow 3$

Homomorphic Representation

People (Nicknames):

$$michael \leftrightarrow$$
$$mike \leftrightarrow$$



Arithmetic:

$$plus(s(0), s(0)) \leftrightarrow$$
$$times(s(s(0)), s(0)) \leftrightarrow \quad 2$$
$$s(s(0)) \leftrightarrow$$

Two Approaches

Approach 1 - Equality:

$$mike = michael$$
$$father(michael) = william$$

***Equivalence of terms** in Herbrand universe*

Approach 2 - Evaluable Terms (nicknames):

$$s(0) + s(0) \text{ --> } s(s(0))$$
$$s(s(0)) * s(0) \text{ --> } s(s(0))$$

*Add **new terms** (terms **not** in the Herbrand universe)*
that "evaluate" to terms in Herbrand universe

Equality

Partitioning the Herbrand Universe

People:

michael *mike*	*maureen*	*katherine* *kathy* *kate*	...

Arithmetic:

...	$plus(s(0), s(0))$ $times(s(s(0)), s(0))$ $s(s(0))$	...

Equality

Co-referential terms:

$$plus(s(0), s(0))$$
$$times(s(s(0)), s(0))$$
$$s(s(0))$$

Equality:

$$equal(plus(s(0),s(0)),\ s(s(0)))$$
$$equal(times(s(s(0)), s(0)),\ s(s(0)))$$

Syntactic Sugar

Formal Syntax:

$$\boldsymbol{equal}(plus(s(0),s(0)),\ s(s(0)))$$
$$\boldsymbol{equal}(times(s(s(0)),s(0)),\ s(s(0)))$$

Syntactic Sugar:

$$s(0) + s(0) = s(s(0))$$
$$s(s(0)) * s(0) = s(s(0))$$

What is Needed

(1) Axioms that define properties of equality.

$$michael = mike$$
$$mike = michael$$

(2) Axioms that ensure that any true sentence that mentions a given term is also true of the sentence in which that term is replaced by an equivalent term.

$$older(michael, maureen) \Leftrightarrow older(mike, maureen)$$

Equality Axioms

Reflexivity

$$\forall x.(x=x)$$

Symmetry

$$\forall x.\forall y.(x=y \;\Rightarrow\; y=x)$$

Transitivity

$$\forall x.\forall y.\forall z.(x=y \wedge y=z \Rightarrow x=z)$$

Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise

Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$b = a \Rightarrow a = b$	2 x UE: 4
7.	$a = b$	IE: 6, 1

Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$b = a \Rightarrow a = b$	2 x UE: 4
7.	$a = b$	IE: 6, 1
8.	$a = b \wedge b = c \Rightarrow a = c$	3 x UE: 5

Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \land y = z \Rightarrow x = z)$	Premise
6.	$b = a \Rightarrow a = b$	2 x UE: 4
7.	$a = b$	IE: 6, 1
8.	$a = b \land b = c \Rightarrow a = c$	3 x UE: 5
9.	$a = b \land b = c$	AI: 7, 2
10.	$a = c$	IE: 8, 9

Substitution Problems

Given:

$f(a) = b$

$f(b) = a$

Prove:

$f(f(a)) = a$

Given:

$\forall x.older(father(x), x)$

$father(bob) = art$

Prove:

$older(art, bob)$

Substitution Axioms

Unary Relations

$$\forall x.\forall y.(p(x) \wedge x{=}y \Rightarrow p(y))$$

Binary Relations

$$\forall u.\forall v.\forall x.\forall y.(q(u,v) \wedge u{=}x \wedge v{=}y \Rightarrow q(x,y))$$

Unary Functions

$$\forall x.\forall y.\forall z.(f(x){=}z \wedge x{=}y \Rightarrow f(y){=}z)$$

Binary Functions

$$\forall u.\forall v.\forall x.\forall y.\forall z.(g(u,v){=}z \wedge u{=}x \wedge v{=}y \Rightarrow g(x,y){=}z)$$

Substitution Proof

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$\forall x.\forall y.\forall z.(f(x)=z \wedge x=y \Rightarrow f(y)=z)$	Premise

Substitution Proof

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \land y = z \Rightarrow x = z)$	Premise
6.	$\forall x.\forall y.\forall z.(f(x)=z \land x=y \Rightarrow f(y)=z)$	Premise
7.	$f(b)=a \land b=f(a) \Rightarrow f(f(a))=a$	3 x UE: 6
8.	$f(a)=b \Rightarrow b=f(a)$	2 x UE: 4
9.	$b=f(a)$	IE: 8, 1

Substitution Proof

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
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7.	$f(b)=a \wedge b=f(a) \Rightarrow f(f(a))=a$	3 x UE: 6
8.	$f(a)=b \Rightarrow b=f(a)$	2 x UE: 4
9.	$b=f(a)$	IE: 8, 1
10.	$f(b) = a \wedge b=f(a)$	AI: 2, 9
11.	$f(f(a))=a$	IE: 7, 10

Too tedious. Too long.

Rules of Inference

Equality Introduction (QI):

$$\frac{}{\tau = \tau}$$

where τ is a ground term

Equality Elimination (QE) - also called *paramodulation*:

$$\frac{\begin{array}{c}\phi \\ \sigma = \tau\end{array}}{\phi_{\sigma \leftarrow \tau}}$$

where σ and τ are ground terms

NB: $\phi_{\sigma \leftarrow \tau}$ is a copy of ϕ with *0 or more* occurrences of σ replaced by τ.

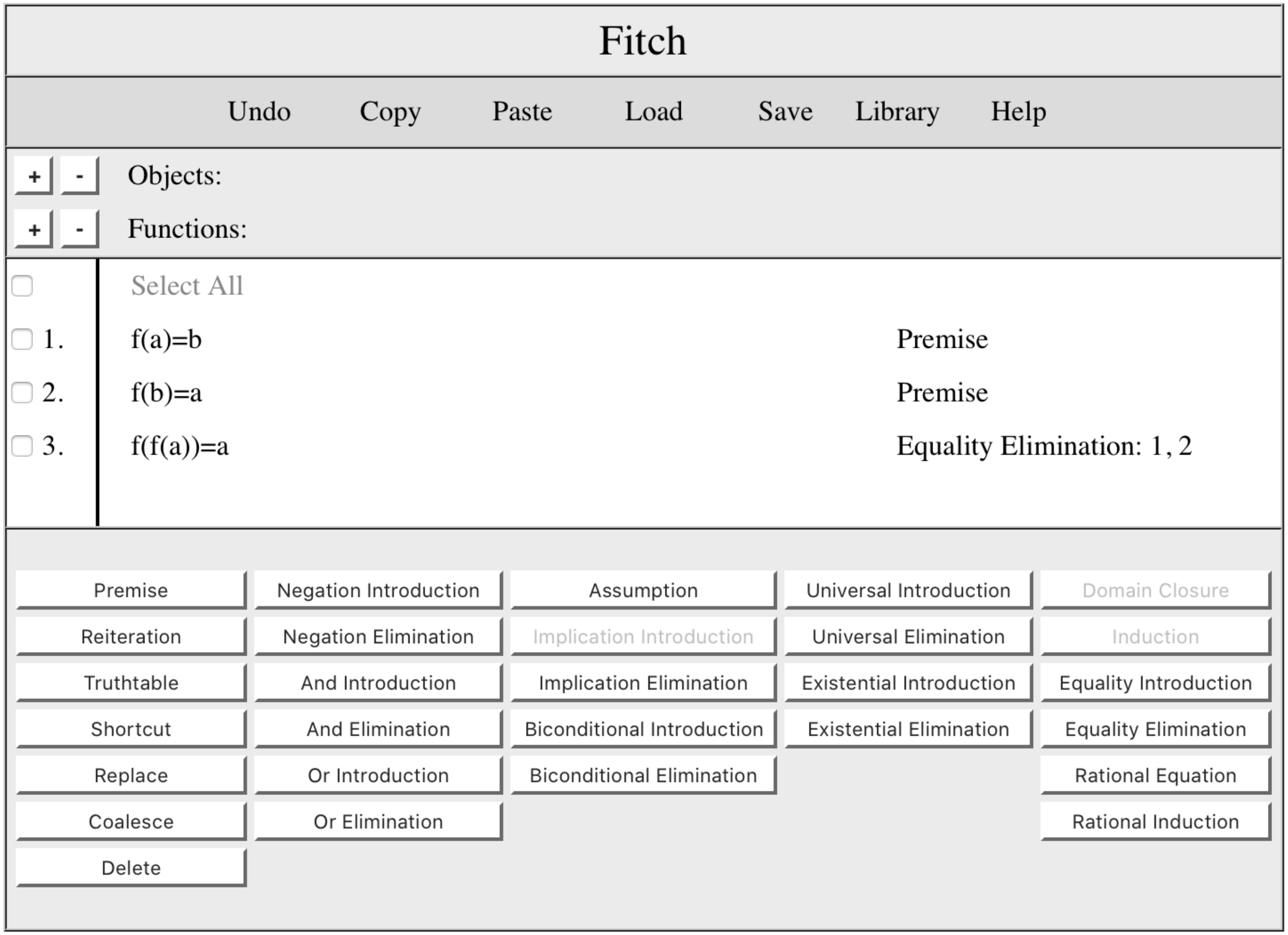
NB: Works using the equality in the opposite direction as well.

Substitution Proof

1. $f(a) = b$	Premise
2. $f(b) = a$	Premise
3. $f(f(a)) = a$	QE: 2, 1

Substitution Proof

1.	$\forall x.older(father(x),x)$	Premise
2.	$father(bob) = art$	Premise
3.	$older(father(bob),bob)$	UE:1
4.	$older(art,bob)$	QE: 3, 2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Objects:
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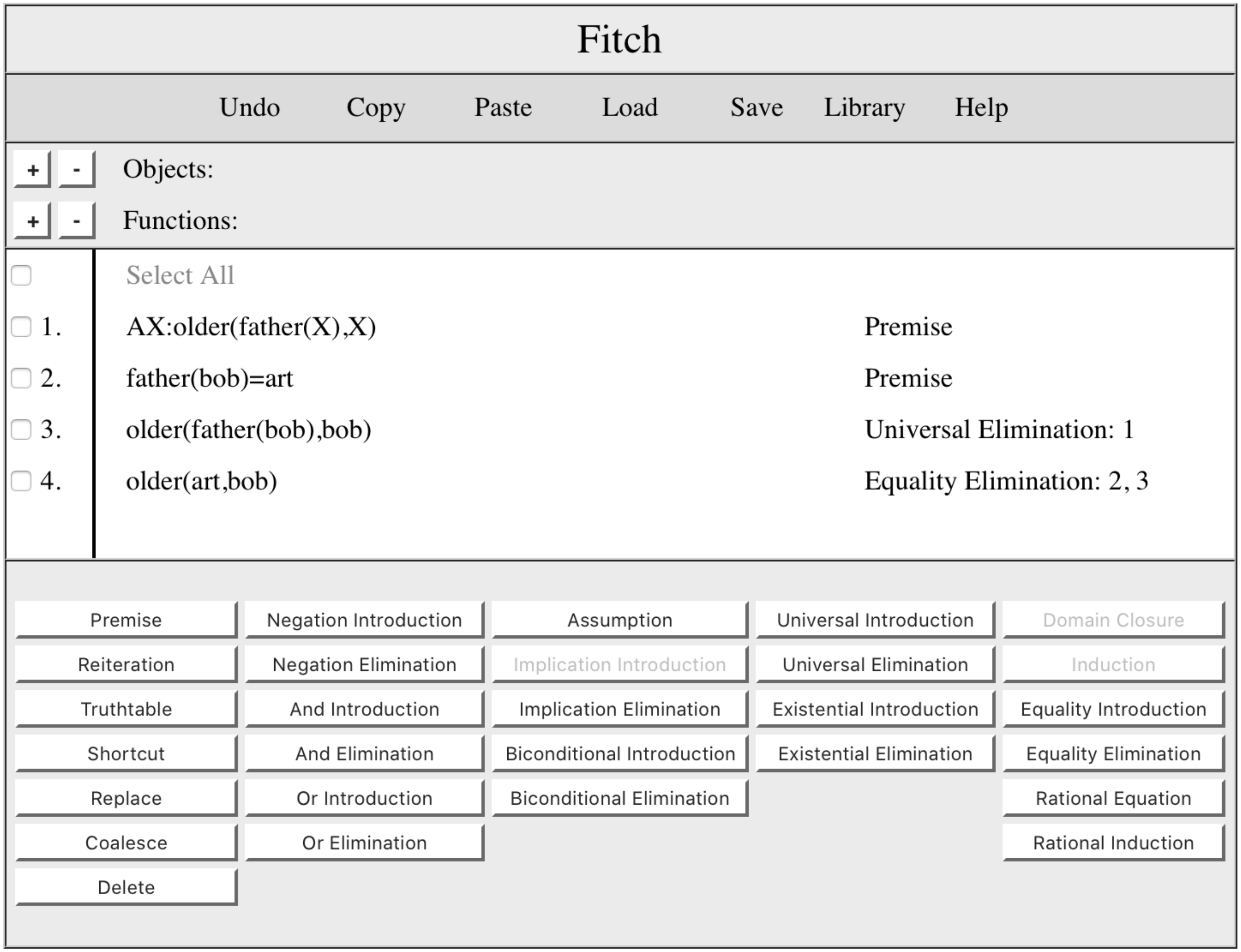
Functions:
Select All
1. f(a)=b Premise
2. f(b)=a Premise
3. f(f(a))=a Equality Elimination: 1, 2
Premise
Negation Introduction
Assumption
Universal Introduction
Domain Closure
Reiteration
Negation Elimination
Implication Introduction
Universal Elimination
Induction
Truthtable
And Introduction
Implication Elimination
Existential Introduction
Equality Introduction
Shortcut
And Elimination
Biconditional Introduction
Existential Elimination
Equality Elimination
Replace
Or Introduction
Biconditional Elimination
Rational Equation
Coalesce
Or Elimination
Rational Induction
Delete


Fitch
Undo
Copy
Paste
Load
Save
Library
Help
Objects:
Functions:
Select All
1. AX:older(father(X),X) Premise
2. father(bob)=art Premise
3. older(father(bob),bob) Universal Elimination: 1
4. older(art,bob) Equality Elimination: 2, 3
Premise
Negation Introduction
Assumption
Universal Introduction
Domain Closure
Reiteration
Negation Elimination
Implication Introduction
Universal Elimination
Induction
Truthtable
And Introduction
Implication Elimination
Existential Introduction
Equality Introduction
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Biconditional Introduction
Existential Elimination
Equality Elimination
Replace
Or Introduction
Biconditional Elimination
Rational Equation
Coalesce
Or Elimination
Rational Induction
Delete

Evaluable Terms

Signature

Object Constant: 0, ...

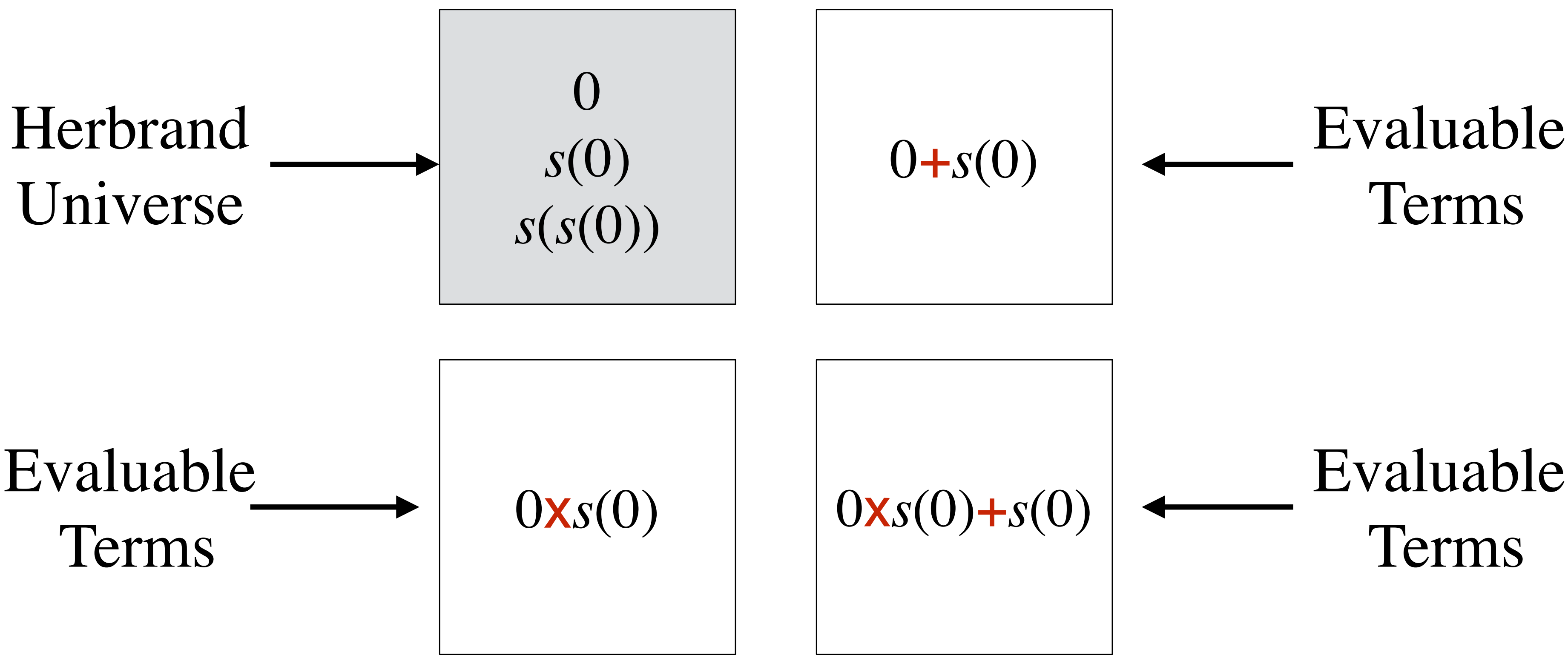
Unary Function Constant: *s*

Evaluable Function Constants: **+**, **x**

Binary Relation Constant: =

*Evaluable functions do **not** expand Herbrand universe.*
*Evaluable terms **refer** to terms in Herbrand universe.*
*Evaluable functions **defined** in terms of other concepts.*

Different Types of Terms



Significance

For induction to work, we must show that a property holds of **all terms in the Herbrand Universe** (Herbrand terms).

When some objects or functions are defined in terms of others, it is not necessary to show that a property holds of **evaluable terms** (since all sentences involving those terms necessarily have the same truth values as sentences with equivalent **Herbrand terms)**.

Upshot: Need to do induction only on objects and functions comprising the Herbrand universe.

Binary Function Example

Object Constant: 0, ...

Unary Function Constant: *s*

Binary Evaluable Function Constant: +

Binary Relation Constant: =

Binary Function Problem

Axioms for +:

$$\forall y.(y + 0 = y)$$
$$\forall x.\forall y.(x + s(y) = s(x + y))$$

Problem: Prove that 0 is a left identity for +.

$$\forall y.(0 + y = y)$$

Inductive Proof

1	$\forall x.\ (x + 0 = x)$	Premise
2.	$\forall x. \forall y.\ (x + s(y) = s(x + y))$	Premise

Inductive Proof

1	$\forall x. (x + 0 = x)$	Premise
2.	$\forall x. \forall y. (x + s(y) = s(x + y))$	Premise
3.	$0 + 0 = 0$	UE: 1

Inductive Proof

1	$\forall x.\,(x + 0 = x)$	Premise
2.	$\forall x.\forall y.\,(x + s(y) = s(x + y))$	Premise
3.	$0 + 0 = 0$	UE: 1
4.	$\vert\, 0 + c = c$	Assumption

Inductive Proof

1	$\forall x. (x + 0 = x)$	Premise
2.	$\forall x. \forall y. (x + s(y) = s(x + y))$	Premise
3.	$0 + 0 = 0$	UE: 1
4.	$\quad \mid\ 0 + c = c$	Assumption
5.	$\quad \mid\ \forall y. (0 + s(y) = s(0 + y))$	UE: 2

Inductive Proof

1	$\forall x. (x + 0 = x)$	Premise
2.	$\forall x. \forall y. (x + s(y) = s(x + y))$	Premise
3.	$0 + 0 = 0$	UE: 1
4.	$\quad 0 + c = c$	Assumption
5.	$\quad \forall y. (0 + s(y) = s(0 + y))$	UE: 2
6.	$\quad 0 + s(c) = s(0 + c)$	UE: 5

Inductive Proof

1	$\forall x. (x + 0 = x)$	Premise
2.	$\forall x. \forall y. (x + s(y) = s(x + y))$	Premise
3.	$0 + 0 = 0$	UE: 1
4.	$\quad 0 + c = c$	Assumption
5.	$\quad \forall y. (0 + s(y) = s(0 + y))$	UE: 2
6.	$\quad 0 + s(c) = s(0 + c)$	UE: 5
7.	$\quad 0 + s(c) = s(c)$	QE: 6, 4

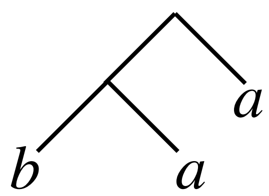
Inductive Proof

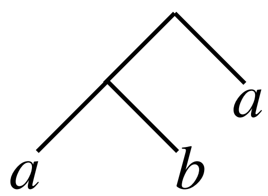
1	$\forall x. (x + 0 = x)$	Premise
2.	$\forall x.\forall y. (x + s(y) = s(x + y))$	Premise
3.	$0 + 0 = 0$	UE: 1
4.	$\quad 0 + c = c$	Assumption
5.	$\quad \forall y. (0 + s(y) = s(0 + y))$	UE: 2
6.	$\quad 0 + s(c) = s(0 + c)$	UE: 5
7.	$\quad 0 + s(c) = s(c)$	QE: 6, 4
8.	$0 + c = c \Rightarrow 0 + s(c) = s(c)$	II: 4, 7

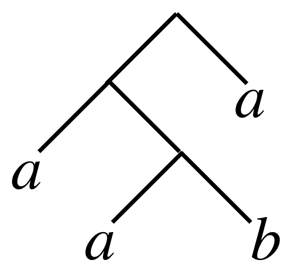
Inductive Proof

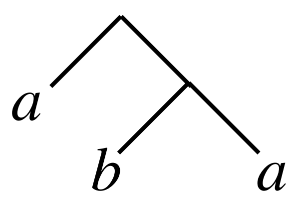
1	$\forall x. (x + 0 = x)$	Premise
2.	$\forall x. \forall y. (x + s(y) = s(x + y))$	Premise
3.	$0 + 0 = 0$	UE: 1
4.	$\quad 0 + c = c$	Assumption
5.	$\quad \forall y. (0 + s(y) = s(0 + y))$	UE: 2
6.	$\quad 0 + s(c) = s(0 + c)$	UE: 5
7.	$\quad 0 + s(c) = s(c)$	QE: 6, 4
8.	$0 + c = c \Rightarrow 0 + s(c) = s(c)$	II: 4, 7
9.	$\forall y. (0 + y = y \Rightarrow 0 + s(y) = s(y))$	UI: 8
10.	$\forall y. (0 + y = y)$	Ind: 3, 9

Trees





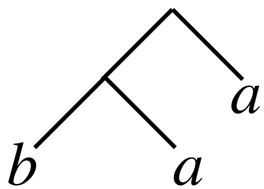




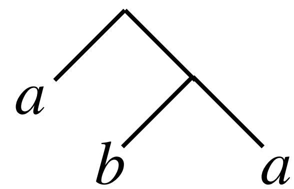
Tree Vocabulary

Object constants: *a*, *b*

Binary function constant: *cons*



$cons(cons(b,a),a)$



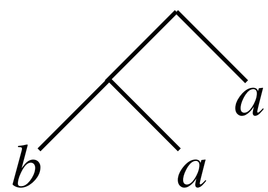
$cons(a,cons(b,a))$

Binary relation constant: *equal*

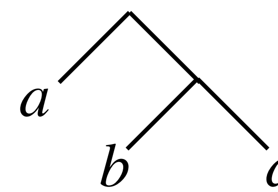
Tree Vocabulary

Object constants: *a*, *b*

Binary function constant: *cons*



$cons(cons(b,a),a)$



$cons(a,cons(b,a))$

Unary evaluable function: *rev*

Binary relation constant: *equal*

Reversing Trees

Reversing Atomic Trees:

$$rev(a) = a$$
$$rev(b) = b$$

Reversing Compound Trees:

$$\forall x.(rev(cons(x, y)) = cons(rev(y), rev(x)))$$

Problem

Reverse is its own inverse. (In other words, the result of reversing a list twice is equal to the original list.)

$$\forall x . rev(rev(x)) = x$$

Let's prove it using induction.

Hint: *rev* is defined in terms of *cons* so we just need to do induction on *cons*.

Proof By Induction

1.	$rev(a) = a$	Premise
2.	$rev(b) = b$	Premise
3.	$\forall x.\forall y.(rev(cons(x, y)) = cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2

Proof By Induction

1.	$rev(a) = a$	Premise
2.	$rev(b) = b$	Premise
3.	$\forall x.\forall y.(rev(cons(x, y)) = cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2
6.	$\vert\; rev(rev(c)) = c \wedge rev(rev(d)) = d$	Assumption

Proof By Induction

1.	$rev(a) = a$	Premise
2.	$rev(b) = b$	Premise
3.	$\forall x.\forall y.(rev(cons(x, y)) = cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2
6.	$\quad\vert\; rev(rev(c)) = c \wedge rev(rev(d)) = d$	Assumption
7.	$\quad\vert\; rev(rev(cons(c, d))) = rev(rev(cons(c, d)))$	QI

Proof By Induction

1.	$rev(a) = a$	Premise
2.	$rev(b) = b$	Premise
3.	$\forall x.\forall y.(rev(cons(x, y)) = cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2
6.	$\quad rev(rev(c)) = c \wedge rev(rev(d)) = d$	Assumption
7.	$\quad rev(rev(cons(c, d))) = rev(rev(cons(c, d)))$	QI
8.	$\quad rev(rev(cons(c, d))) = rev(cons(rev(d), rev(c)))$	UE QE: 7, 3

Proof By Induction

1.	$rev(a) = a$	Premise
2.	$rev(b) = b$	Premise
3.	$\forall x.\forall y.(rev(cons(x, y)) = cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2
6.	| $rev(rev(c)) = c \wedge rev(rev(d)) = d$	Assumption
7.	| $rev(rev(cons(c, d))) = rev(rev(cons(c, d)))$	QI
8.	| $rev(rev(cons(c, d))) = rev(cons(rev(d), rev(c)))$	UE QE: 7, 3
9.	| $rev(rev(cons(c, d))) = cons(rev(rev(c)), rev(rev(d)))$	UE QE: 8, 3

Proof By Induction

1.	$rev(a) = a$	Premise
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3.	$\forall x. \forall y. (rev(cons(x, y)) = cons(rev(y), rev(x)))$	Premise
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8.	$\quad rev(rev(cons(c, d))) = rev(cons(rev(d), rev(c)))$	UE QE: 7, 3
9.	$\quad rev(rev(cons(c, d))) = cons(rev(rev(c)), rev(rev(d)))$	UE QE: 8, 3
10.	$\quad rev(rev(cons(c, d))) = cons(c, d)$	2 x QE: 9, 6

Proof By Induction

1.	$rev(a) = a$	Premise
2.	$rev(b) = b$	Premise
3.	$\forall x.\forall y.(rev(cons(x, y))= cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2
6.	$\mid\ rev(rev(c)) = c \wedge rev(rev(d)) = d$	Assumption
7.	$\mid\ rev(rev(cons(c, d))) = rev(rev(cons(c, d)))$	QI
8.	$\mid\ rev(rev(cons(c, d))) = rev(cons(rev(d), rev(c)))$	UE QE: 7, 3
9.	$\mid\ rev(rev(cons(c, d))) = cons(rev(rev(c)), rev(rev(d)))$	UE QE: 8, 3
10.	$\mid\ rev(rev(cons(c, d))) = cons(c, d)$	2 x QE: 9, 6
11.	$rev(rev(c))=c \wedge rev(rev(d))=d \Rightarrow rev(rev(cons(c, d)))=cons(c, d)$	II: 6, 10

Proof By Induction

1.	$rev(a) = a$	Premise
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3.	$\forall x.\forall y.(rev(cons(x, y)) = cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2
6.	$\quad rev(rev(c)) = c \wedge rev(rev(d)) = d$	Assumption
7.	$\quad rev(rev(cons(c, d))) = rev(rev(cons(c, d)))$	QI
8.	$\quad rev(rev(cons(c, d))) = rev(cons(rev(d), rev(c)))$	UE QE: 7, 3
9.	$\quad rev(rev(cons(c, d))) = cons(rev(rev(c)), rev(rev(d)))$	UE QE: 8, 3
10.	$\quad rev(rev(cons(c, d))) = cons(c, d)$	2 x QE: 9, 6
11.	$rev(rev(c))=c \wedge rev(rev(d))=d \Rightarrow rev(rev(cons(c, d)))=cons(c, d)$	II: 6, 10
12.	$\forall x.\forall y.(rev(rev(x)) = x \wedge rev(rev(y)) = y \Rightarrow rev(rev(cons(x, y))) = cons(x\ y))$	2 x UI: 11

Proof By Induction

1.	$rev(a) = a$	Premise
2.	$rev(b) = b$	Premise
3.	$\forall x.\forall y.(rev(cons(x, y))= cons(rev(y), rev(x)))$	Premise
4.	$rev(rev(a)) = a$	QE: 1, 1
5.	$rev(rev(b)) = b$	QE: 2, 2
6.	$\quad rev(rev(c)) = c \wedge rev(rev(d)) = d$	Assumption
7.	$\quad rev(rev(cons(c, d))) = rev(rev(cons(c, d)))$	QI
8.	$\quad rev(rev(cons(c, d))) = rev(cons(rev(d), rev(c)))$	UE QE: 7, 3
9.	$\quad rev(rev(cons(c, d))) = cons(rev(rev(c)), rev(rev(d))))$	UE QE: 8, 3
10.	$\quad rev(rev(cons(c, d))) = cons(c, d))$	2 x QE: 9, 6
11.	$rev(rev(c))=c \wedge rev(rev(d))=d \Rightarrow rev(rev(cons(c, d)))=cons(c, d)$	II: 6, 10
12.	$\forall x.\forall y.(rev(rev(x)) = x \wedge rev(rev(y)) = y \Rightarrow$ $rev(rev(cons(x, y))) = cons(x\ y))$	2 x UI: 11
13.	$\forall x:(rev(rev(x)) = x)$	Ind: 4, 5, 12

Polynomial Arithmetic

Signature for Arithmetic

Object Constant: 0

Unary Function Constant: *s*

Binary Evaluable Function Constants:

plus - addition

times - multiplication

Binary Relation Constant: =

More Syntactic Sugar

Formal Syntax:

$$\textbf{\textit{equal}}(plus(s(0),s(0)),\ s(s(0)))$$
$$\textbf{\textit{equal}}(times(s(s(0)),s(0)),\ s(s(0)))$$

Syntactic Sugar:

$$1 + 1 = 2$$
$$2 \times 1 = 2$$

Definitions of Evaluable Arithmetic Functions

Addition:

$$\forall x.(x + 0 = x)$$

$$\forall x.\forall y.(x + (y + 1) = (x + y) + 1)$$

Multiplication:

$$\forall y.(0 \times y = 0)$$

$$\forall x.\forall y.((x + 1) \times y = (x \times y) + y)$$

Equality Proof

1.	$b = a$	Premise
2.	$b = c$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise

Long Messy Proofs

1.	$f(a) = b$	Premise
2.	$f(b) = a$	Premise
3.	$\forall x.(x = x)$	Premise
4.	$\forall x.\forall y.(x = y \Rightarrow y = x)$	Premise
5.	$\forall x.\forall y.\forall z.(x = y \wedge y = z \Rightarrow x = z)$	Premise
6.	$\forall x.\forall y.\forall z.(f(x)=z \wedge x=y \Rightarrow f(y)=z)$	Premise
7.	$f(b)=a \wedge b=f(a) \Rightarrow f(f(a))=a$	3 x UE: 6
8.	$f(a)=b \Rightarrow b=f(a)$	2 x UE: 4
9.	$b=f(a)$	IE: 8, 1
10.	$f(b) = a \wedge b=f(a)$	AI: 2, 9
11.	$f(f(a))=a$	IE: 7, 10

Rational Equation

Rule of Inference (Rational Equation):

$$\frac{}{\sigma = \tau}$$

*where σ and τ are **equivalent** polynomials*

Example:

$$\frac{}{(c+1)*(c+1) = c*c+2*c+1}$$

Rational Induction

+ definable in terms of s, so we need only do induction on s.

$$\frac{\phi[0] \qquad \forall x.(\phi[x] \Rightarrow \phi[s(x)])}{\forall x.\phi[x]}$$

Rational Induction (since $x + 1 = s(x)$)

$$\frac{\phi[0] \qquad \forall x.(\phi[x] \Rightarrow \phi[x + 1])}{\forall x.\phi[x]}$$

Problem

Consider the following function.

$$f(0) = 0$$
$$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$$

$f(z)$ is 2 x the sum of numbers from 0 through z.

Problem

Function definition:

$$f(0) = 0$$
$$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$$

Claim: f can be defined *non-recursively* in terms of + and x.

$$\forall z.(f(z) = z \times (z + 1))$$

Let's prove it, using induction.

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 = 0 \times (0 + 1)$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1

Base Case for :

$$\forall z.(f(z) = z \times (z + 1))$$

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$\mid f(c) = c \times (c+1)$	Assumption

We want to prove:

$\forall z.(f(z) = z \times (z + 1) \Rightarrow f(z+1) = (z + 1) \times ((z + 1) + 1))$

To do that, we need to prove:

$f(c) = c \times (c + 1) \Rightarrow f(c+1) = (c + 1) \times ((c + 1) + 1)$

To do that, we assume $f(c) = c \times (c + 1)$

and prove $f(c+1) = (c + 1) \times ((c + 1) + 1)$

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$\quad \mid f(c) = c \times (c+1)$	Assumption
6.	$\quad \mid f(c+1) = f(c) + (c+1) + (c+1)$	UE: 2

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$\quad\vert\ f(c) = c \times (c+1)$	Assumption
6.	$\quad\vert\ f(c+1) = f(c) + (c+1) + (c+1)$	UE: 2
7.	$\quad\vert\ f(c+1) = c \times (c+1) + (c+1) + (c+1)$	QE: 5, 6

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$\mid f(c) = c \times (c+1)$	Assumption
6.	$\mid f(c+1) = f(c) + (c+1) + (c+1)$	UE: 2
7.	$\mid f(c+1) = c \times (c+1) + (c+1) + (c+1)$	QE: 5, 6
8.	$\mid c \times (c+1) + (c+1) + (c+1) = (c+1) \times ((c+1) + 1)$	Equation

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$f(c) = c \times (c+1)$	Assumption
6.	$f(c+1) = f(c) + (c+1) + (c+1)$	UE: 2
7.	$f(c+1) = c \times (c+1) + (c+1) + (c+1)$	QE: 5, 6
8.	$c \times (c+1) + (c+1) + (c+1) = (c+1) \times ((c+1) + 1)$	Equation
9.	$f(c+1) = (c+1) \times ((c+1) + 1)$	QE: 8, 7

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$\quad f(c) = c \times (c+1)$	Assumption
6.	$\quad f(c+1) = f(c) + (c+1) + (c+1)$	UE: 2
7.	$\quad f(c+1) = c \times (c+1) + (c+1) + (c+1)$	QE: 5, 6
8.	$\quad c \times (c+1) + (c+1) + (c+1) = (c+1) \times ((c+1) + 1)$	Equation
9.	$\quad f(c+1) = (c+1) \times ((c+1) + 1)$	QE: 8, 7
10.	$f(c) = c \times (c+1) \Rightarrow f(c+1) = (c+1) \times ((c+1) + 1)$	II: 5, 9

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$\quad\vert\ f(c) = c \times (c+1)$	Assumption
6.	$\quad\vert\ f(c+1) = f(c) + (c+1) + (c+1)$	UE: 2
7.	$\quad\vert\ f(c+1) = c \times (c+1) + (c+1) + (c+1)$	QE: 5, 6
8.	$\quad\vert\ c \times (c+1) + (c+1) + (c+1) = (c+1) \times ((c+1) + 1)$	Equation
9.	$\quad\vert\ f(c+1) = (c+1) \times ((c+1) + 1)$	QE: 8, 7
10.	$f(c) = c \times (c+1) \Rightarrow f(c+1) = (c+1) \times ((c+1) + 1)$	II: 5, 9
11.	$\forall z.(f(z) = z \times (z+1) \Rightarrow f(z+1) = (z+1) \times ((z+1) + 1))$	UI: 10

Problem

1	$f(0) = 0$	Premise
2.	$\forall z.(f(z+1) = f(z) + (z+1) + (z+1))$	Premise
3.	$0 \times (0 + 1) = 0$	Equation
4.	$f(0) = 0 \times (0 + 1)$	QE: 3, 1
5.	$\quad f(c) = c \times (c+1)$	Assumption
6.	$\quad f(c+1) = f(c) + (c+1) + (c+1)$	UE: 2
7.	$\quad f(c+1) = c \times (c+1) + (c+1) + (c+1)$	QE: 5, 6
8.	$\quad c \times (c+1) + (c+1) + (c+1) = (c+1) \times ((c+1) + 1)$	Equation
9.	$\quad f(c+1) = (c+1) \times ((c+1) + 1)$	QE: 8, 7
10.	$f(c) = c \times (c+1) \Rightarrow f(c+1) = (c+1) \times ((c+1) + 1)$	II: 5, 9
11.	$\forall z.(f(z) = z \times (z+1) \Rightarrow f(z+1) = (z+1) \times ((z+1) + 1))$	UI: 10
12.	$\forall z.(f(z) = z \times (z+1))$	Ratind: 4, 11

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Objects:

Functions:

Select All

1.	f(0)=0	Premise
2.	AZ:f(Z+1)=f(Z)+((Z+1)+(Z+1))	Premise
3.	0*(0+1)=0	Equation
4.	f(0)=0*(0+1)	Equality Elimination: 3, 1
5.	f(c)=c*(c+1)	Assumption
6.	f(c+1)=f(c)+((c+1)+(c+1))	Universal Elimination: 2
7.	f(c+1)=c*(c+1)+((c+1)+(c+1))	Equality Elimination: 5, 6
8.	c*(c+1)+((c+1)+(c+1))=(c+1)*((c+1)+1)	Equation
9.	f(c+1)=(c+1)*((c+1)+1)	Equality Elimination: 8, 7
10.	f(c)=c*(c+1) => f(c+1)=(c+1)*((c+1)+1)	Implication Introduction: 5, 9
11.	AZ:(f(Z)=Z*(Z+1) => f(Z+1)=(Z+1)*((Z+1)+1))	Universal Introduction: 10
12.	AZ:f(Z)=Z*(Z+1)	Rational Induction
Goal	AZ:f(Z)=Z*(Z+1)	Complete

Yet Another Problem

Recursive Function:

$g(0) = 0$

$g(z + 1) = g(z) + (2 \times z + 1)$

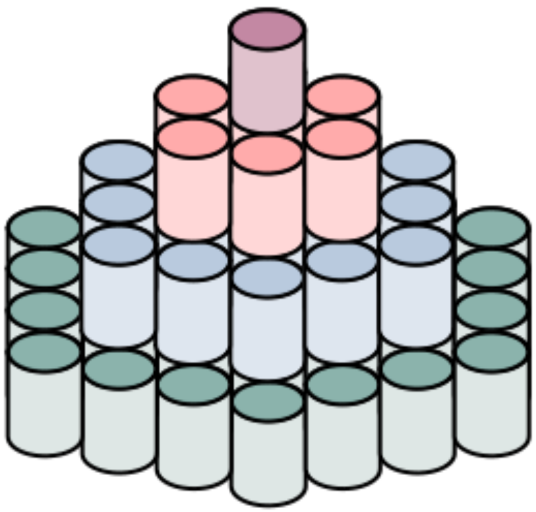
z	$g(z)$
0	0
1	1
2	4
3	9

Non-recursive definition in terms of z:

$g(z) = z \times z$

And Another

Number of cans per layer (starting at 1):



$h(1) = 1$

$h(z + 1) = h(z) + 6 \times z$

Non-recursive definition in terms of + and x:

$h(z+1) = 3 \times z \times (z + 1) + 1$