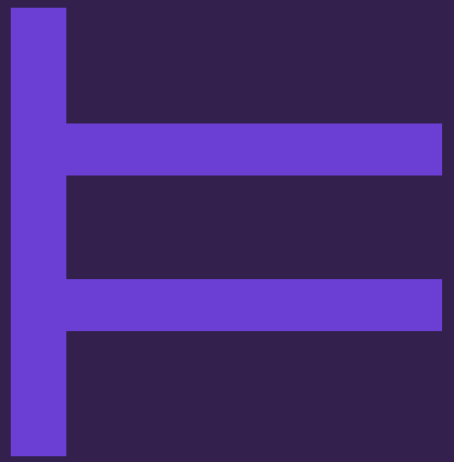
CS 157 Quiz 1 Review

Propositional logic: semantics, proof systems, and resolution



Agenda

1

Validity, unsatisfiability, contingency

Classifying a single sentence

2

Entailment, equivalence, consistency

How sentences relate to each other

3

Soundness and completeness

What a proof system guarantees

4

Hilbert proofs

One rule and three axiom schemata

5

Natural deduction and Fitch proofs

Subproofs and ten core rules

6

Resolution

Clausal form and refutation

01

Validity, unsatisfiability, contingency

Is a sentence always true, never true, or somewhere in between?

Validity, unsatisfiability, contingency

Propositional logic and analysis

Valid

Satisfied by every truth assignment.

$$p \vee \neg p$$

Always yields T in a truth table.

Unsatisfiable

Not satisfied by any truth assignment.

$$p \wedge \neg p$$

Always yields F in a truth table.

Contingent

Some truth assignment satisfies it, and some truth assignment falsifies it.

$$p \wedge q$$

True if p and q are both true; false if both are false.

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

b. $p \wedge (p \Rightarrow \neg q) \wedge q$

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

b. $p \wedge (p \Rightarrow \neg q) \wedge q$

Reasoning

$p \Rightarrow \neg q \equiv \neg p \vee \neg q$

So the whole formula $\equiv p \wedge (\neg p \vee \neg q) \wedge q$

$\equiv p \wedge \neg q \wedge q$

This is a contradiction.

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

b. $p \wedge (p \Rightarrow \neg q) \wedge q$

Reasoning

$p \Rightarrow \neg q \equiv \neg p \vee \neg q$

So the whole formula $\equiv p \wedge (\neg p \vee \neg q) \wedge q$

$\equiv p \wedge \neg q \wedge q$

This is a contradiction.

Answer Valid **Unsatisfiable** Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

g. $((\neg p \Rightarrow q) \Rightarrow (\neg q \Rightarrow p)) \wedge (p \vee q)$

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

g. $((\neg p \Rightarrow q) \Rightarrow (\neg q \Rightarrow p)) \wedge (p \vee q)$

Reasoning

$\neg p \Rightarrow q \equiv p \vee q$ and $\neg q \Rightarrow p \equiv p \vee q$

So the first part is $(p \vee q) \Rightarrow (p \vee q) \equiv \top$

The formula reduces to $p \vee q$

True when $p = 1, q = 0$; false when $p = 0, q = 0$.

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

g. $((\neg p \Rightarrow q) \Rightarrow (\neg q \Rightarrow p)) \wedge (p \vee q)$

Reasoning

$\neg p \Rightarrow q \equiv p \vee q$ and $\neg q \Rightarrow p \equiv p \vee q$

So the first part is $(p \vee q) \Rightarrow (p \vee q) \equiv \top$

The formula reduces to $p \vee q$

True when $p = 1, q = 0$; false when $p = 0, q = 0$.

Answer Valid | Unsatisfiable | **Contingent**

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

h. $(\neg p \vee q) \Rightarrow (q \wedge (p \Leftrightarrow q))$

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

h. $(\neg p \vee q) \Rightarrow (q \wedge (p \Leftrightarrow q))$

Reasoning

Falsified by p = 0, q = 0 (antecedent true, consequent false)

Satisfied by p = 1, q = 1 (both parts true)

Also satisfied by p = 1, q = 0 (antecedent false)

So it is neither a tautology nor a contradiction.

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

h. $(\neg p \vee q) \Rightarrow (q \wedge (p \Leftrightarrow q))$

Reasoning

Falsified by p = 0, q = 0 (antecedent true, consequent false)

Satisfied by p = 1, q = 1 (both parts true)

Also satisfied by p = 1, q = 0 (antecedent false)

So it is neither a tautology nor a contradiction.

Answer Valid | Unsatisfiable | **Contingent**

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

i. $((\neg r \Rightarrow \neg p \wedge \neg q) \vee s) \Leftrightarrow ((p \vee q) \Rightarrow (r \vee s))$

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

i. $((\neg r \Rightarrow \neg p \wedge \neg q) \vee s) \Leftrightarrow ((p \vee q) \Rightarrow (r \vee s))$

Reasoning

LHS $\equiv (r \vee (\neg p \wedge \neg q)) \vee s \equiv (r \vee s) \vee (\neg p \wedge \neg q)$

RHS $\equiv \neg(p \vee q) \vee (r \vee s) \equiv (\neg p \wedge \neg q) \vee (r \vee s)$

Both sides are identical, so the biconditional is a tautology.

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

i. $((\neg r \Rightarrow \neg p \wedge \neg q) \vee s) \Leftrightarrow ((p \vee q) \Rightarrow (r \vee s))$

Reasoning

LHS $\equiv (r \vee (\neg p \wedge \neg q)) \vee s \equiv (r \vee s) \vee (\neg p \wedge \neg q)$

RHS $\equiv \neg(p \vee q) \vee (r \vee s) \equiv (\neg p \wedge \neg q) \vee (r \vee s)$

Both sides are identical, so the biconditional is a tautology.

Answer **Valid** Unsatisfiable Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

j. $(p \wedge (q \Rightarrow r)) \Leftrightarrow ((\neg p \vee q) \Rightarrow (p \wedge r))$

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

j. $(p \wedge (q \Rightarrow r)) \Leftrightarrow ((\neg p \vee q) \Rightarrow (p \wedge r))$

Reasoning

LHS $\equiv p \wedge (\neg q \vee r)$

RHS $\equiv \neg(\neg p \vee q) \vee (p \wedge r)$

$\equiv (p \wedge \neg q) \vee (p \wedge r) \equiv p \wedge (\neg q \vee r)$

The sides match, so the biconditional is a tautology.

Answer Valid | Unsatisfiable | Contingent

Practice: valid, unsatisfiable, or contingent?

Determine whether the following formula is valid, unsatisfiable, or contingent.

j. $(p \wedge (q \Rightarrow r)) \Leftrightarrow ((\neg p \vee q) \Rightarrow (p \wedge r))$

Reasoning

LHS $\equiv p \wedge (\neg q \vee r)$

RHS $\equiv \neg(\neg p \vee q) \vee (p \wedge r)$

$\equiv (p \wedge \neg q) \vee (p \wedge r) \equiv p \wedge (\neg q \vee r)$

The sides match, so the biconditional is a tautology.

Answer **Valid** Unsatisfiable Contingent

02

Entailment, equivalence, consistency

How sets of sentences relate to one another

Logical entailment

Definition

A set of sentences logically entails a sentence iff every truth assignment that satisfies the premises also satisfies the conclusion.

Example 1

$$\{p \Rightarrow r\} \models (p \Rightarrow q \vee r)$$

Example 2

$$\{p \Rightarrow q \vee r,\ p \Rightarrow r\} \not\models (q \Rightarrow r)$$

Extra practice: introlic.stanford.edu/exercises/exercise_03_03.html · introlic.stanford.edu/exercises/exercise_03_04.html

Practice: is this rule logically correct?

a

All x are z.

All y are z.

Therefore, some x are y.

Answer

Correct

Incorrect

Practice: is this rule logically correct?

a

All x are z.

All y are z.

Therefore, some x are y.

Answer

Correct

Incorrect

Why

Sharing a property does not make two groups overlap. All cats are animals and all dogs are animals, yet no cats are dogs.

Practice: is this rule logically correct?

b

Some x are y.

All y are z.

Therefore, some x are z.

Answer

Correct

Incorrect

Practice: is this rule logically correct?

b

Some x are y.

All y are z.

Therefore, some x are z.

Answer

Correct

Incorrect

Why

Take any x that is also a y. Since all y are z, that same x is a z. So some x are z.

Practice: is this rule logically correct?

C

All x are y.

Some y are z.

Therefore, some x are z.

Answer

Correct

Incorrect

Practice: is this rule logically correct?

C

All x are y.

Some y are z.

Therefore, some x are z.

Answer

Correct

Incorrect

Why

The y's that are z need not be x's. All cats are animals and some animals are dogs, yet no cats are dogs.

Logical equivalence

Definition

φ and ψ are logically equivalent if they entail each other.

Equivalence Theorem: φ and ψ are equivalent iff (φ ⇔ ψ) is valid.

Example 1: equivalent

$$((p \Rightarrow q) \vee (q \Rightarrow r)) \text{ and } (p \vee \neg p)$$

Example 2: not equivalent

$$(p \wedge q \Rightarrow r) \text{ and } (p \wedge r \Rightarrow q)$$

Extra practice: intrologic.stanford.edu/exercises/exercise_03_02.html

Logical consistency

Definition

φ and ψ are consistent if there is a truth assignment that satisfies both.

Example 1: consistent

$\{p \Rightarrow r,\ q \Rightarrow r,\ p \vee q\}$ and r

Example 2: not consistent

$\{p \Rightarrow r,\ q \Rightarrow r,\ p \vee q\}$ and $\neg r$

Extra practice: introlic.stanford.edu/exercises/exercise_03_05.html

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

1. If $\Gamma \models \varphi$ and $\Delta \not\models \varphi$, then $\Gamma \cup \Delta \models \varphi$

Answer True False

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

1. If $\Gamma \models \varphi$ and $\Delta \not\models \varphi$, then $\Gamma \cup \Delta \models \varphi$

Reasoning

If $\Gamma \models \varphi$, then every truth assignment that satisfies Γ satisfies φ.

Any assignment that satisfies $\Gamma \cup \Delta$ also satisfies Γ, so it satisfies φ.

Thus $\Gamma \cup \Delta \models \varphi$. (The fact that $\Delta \not\models \varphi$ is irrelevant.)

Answer True False

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

1. If $\Gamma \models \varphi$ and $\Delta \not\models \varphi$, then $\Gamma \cup \Delta \models \varphi$

Reasoning

If $\Gamma \models \varphi$, then every truth assignment that satisfies Γ satisfies φ.

Any assignment that satisfies $\Gamma \cup \Delta$ also satisfies Γ, so it satisfies φ.

Thus $\Gamma \cup \Delta \models \varphi$. (The fact that $\Delta \not\models \varphi$ is irrelevant.)

Answer **True** False

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

2. If $\Gamma \not\models \psi$, then $\Gamma \models \neg\psi$

Answer True False

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

2. If $\Gamma \nvDash \psi$, then $\Gamma \vDash \neg\psi$

Reasoning

Counterexample: let $\Gamma = \emptyset$ and $\psi = p$.

Then $\Gamma \nvDash \psi$, since some assignments make p false.

But also $\Gamma \nvDash \neg\psi$, since some assignments make p true.

So $\Gamma \vDash \neg\psi$ does not follow.

Answer True False

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

2. If $\Gamma \nvDash \psi$, then $\Gamma \vDash \neg\psi$

Reasoning

Counterexample: let $\Gamma = \emptyset$ and $\psi = p$.

Then $\Gamma \nvDash \psi$, since some assignments make p false.

But also $\Gamma \nvDash \neg\psi$, since some assignments make p true.

So $\Gamma \vDash \neg\psi$ does not follow.

Answer True **False**

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

3. If $\Gamma \models \neg\psi$, then $\Gamma \not\models \psi$

Answer True False

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

3. If $\Gamma \models \neg\psi$, then $\Gamma \not\models \psi$

Reasoning

Counterexample with an inconsistent set: $\Gamma = \{q, \neg q\}$.

Then $\Gamma \models \chi$ for every sentence χ, so both $\Gamma \models \neg\psi$ and $\Gamma \models \psi$.

(If Γ were consistent, the implication would hold.)

Answer True False

Practice: logical entailment with sets

Γ and Δ are sets of sentences in Propositional Logic; φ and ψ are individual sentences. True or false?

3. If $\Gamma \models \neg\psi$, then $\Gamma \not\models \psi$

Reasoning

Counterexample with an inconsistent set: $\Gamma = \{q, \neg q\}$.

Then $\Gamma \models \chi$ for every sentence χ, so both $\Gamma \models \neg\psi$ and $\Gamma \models \psi$.

(If Γ were consistent, the implication would hold.)

Answer True **False**

03

Soundness and completeness

Does what we can prove match what is true?

Soundness and completeness

Soundness

A proof system is sound iff every conclusion that is provable from a set of premises is logically entailed.

$$\text{If } \Delta \vdash \varphi, \text{ then } \Delta \models \varphi$$

“Everything provable is true.”

Completeness

A proof system is complete iff every conclusion that is logically entailed by a set of premises is provable.

$$\text{If } \Delta \models \varphi, \text{ then } \Delta \vdash \varphi$$

“Everything true is provable.”

Are Hilbert and Fitch sound and complete?

	Sound?	Complete?
Hilbert	Yes	Yes
Fitch	Yes	Yes

Both systems prove exactly the conclusions that are logically entailed.

04 Hilbert proofs

One rule of inference and three axiom schemata

Hilbert proofs: axiom schemata and rule instances

Axiom schema

A formula pattern with metavariables.

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

An instance replaces metavariables consistently, for example:

$$p \Rightarrow (q \Rightarrow p)$$

Rule of inference

A pattern leading from premises to conclusion(s).

- Apply a rule only when its premise instances appear as whole, top-level formulas (not inside other connectives).
- Substitutions must be consistent across all occurrences of each metavariable.

The Hilbert system: a single rule

Modus Ponens

Also called Implication Elimination

$$\frac{\varphi \Rightarrow \psi \qquad \varphi}{\psi}$$

For propositional logic with ¬ and ⇒

If you have an implication φ ⇒ ψ and you have φ, you can conclude ψ.

This is the only rule of inference in the Hilbert system. Everything else comes from the axiom schemata.

The Hilbert system: three axiom schemata

Used as theorems on demand

IC **Implication Creation** $\varphi \Rightarrow (\psi \Rightarrow \varphi)$

ID **Implication Distribution** $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$

IR **Implication Reversal** $(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)$

05

Natural deduction and Fitch proofs

Subproofs, ten core rules, and how to pick the next move

Conditional proofs: what they are and why they help

Natural deduction adds **subproofs**: assumptions nested inside the main proof.

- You may assume anything inside a subproof. Those assumptions are local: they cannot be used outside the subproof except to discharge them via a conditional rule.
- Typically, use Implication Introduction to discharge assumptions.

Shape of a conditional proof

1.	φ	Assumption
2.	⋮	
3.	ψ	
4.	$\varphi \Rightarrow \psi$	$\Rightarrow$ Introduction

Conditional proofs: example

Derive **p ⇒ r** from **p ⇒ q** and **q ⇒ r**

1.	p ⇒ q	Premise
2.	q ⇒ r	Premise
3.	▏ p	Assumption
4.	▏ q	Implication Elimination: 3, 1
5.	▏ r	Implication Elimination: 4, 2
6.	p ⇒ r	Implication Introduction: 3, 5

The subproof (lines 3 to 5) shows that assuming p lets us derive r, so we conclude p ⇒ r by Implication Introduction.

The Fitch rules: conjunction

Ten core rules: an introduction rule and an elimination rule for each of the five connectives

And Introduction

$$\frac{\begin{array}{c}\varphi_1 \\ \vdots \\ \varphi_n\end{array}}{\varphi_1 \wedge \ldots \wedge \varphi_n}$$

And Elimination

$$\frac{\varphi_1 \wedge \ldots \wedge \varphi_n}{\varphi_i}$$

The Fitch rules: disjunction and negation

Or Introduction

$$\frac{\varphi_i}{\varphi_1 \vee \ldots \vee \varphi_n}$$

Or Elimination

$$\frac{\varphi_1 \vee \ldots \vee \varphi_n \qquad \varphi_1 \Rightarrow \psi \quad \ldots \quad \varphi_n \Rightarrow \psi}{\psi}$$

Negation Introduction

Proof by contradiction

$$\frac{\varphi \Rightarrow \psi \qquad \varphi \Rightarrow \neg\psi}{\neg\varphi}$$

Negation Elimination

$$\frac{\neg\neg\varphi}{\varphi}$$

The Fitch rules: implication and biconditional

Implication Introduction

The conditional rule: a subproof from φ to ψ

$$\frac{\varphi \vdash \psi}{\varphi \Rightarrow \psi}$$

Implication Elimination

Modus Ponens

$$\frac{\varphi \Rightarrow \psi \qquad \varphi}{\psi}$$

Biconditional Introduction

$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\varphi \Leftrightarrow \psi}$$

Biconditional Elimination

$$\frac{\varphi \Leftrightarrow \psi}{\varphi \Rightarrow \psi \quad \text{and} \quad \psi \Rightarrow \varphi}$$

Fitch tips: let the goal pick the move

Goal	$\varphi \Rightarrow \psi$	Assume φ, derive ψ, then discharge with ⇒I.
Goal	$\varphi \wedge \psi$	Prove φ and ψ separately, then ∧I.
Goal	$\varphi \vee \psi$	Prove either disjunct, then ∨I.
Goal	$\neg\varphi$	Assume φ, derive φ ⇒ ψ and φ ⇒ ¬ψ, then ¬I.

General trick: to prove φ, try assuming ¬φ, derive a contradiction to get ¬¬φ, then use ¬E.

Mini-template for φ ⇒ ψ

1.	q	Premise
2.	| p	Assumption
3.	| q	Reiteration: 1
4.	p ⇒ q	⇒I: 2, 3

Fitch tips: let the premises pick the move

You have

$\varphi \Rightarrow \psi$

You want

ψ

Try

Try proving φ, then use ⇒E.

You have

$\varphi \vee \psi$

You want

χ

Try

Try proving $\varphi \Rightarrow \chi$ and $\psi \Rightarrow \chi$, then use ∨E.

Fitch proof practice 1

Prove the following using Fitch-style natural deduction.

GOAL

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 1 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	**$p \Rightarrow q$**	Assumption

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 2 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$p \Rightarrow q$	Assumption
3.	**p**	Assumption

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 3 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$p \Rightarrow q$	Assumption
3.	p	Assumption
4.	**$q \Rightarrow r$**	Implication Elimination: 1, 3

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 4 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$p \Rightarrow q$	Assumption
3.	p	Assumption
4.	$q \Rightarrow r$	Implication Elimination: 1, 3
5.	**q**	Implication Elimination: 2, 3

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 5 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$p \Rightarrow q$	Assumption
3.	p	Assumption
4.	$q \Rightarrow r$	Implication Elimination: 1, 3
5.	q	Implication Elimination: 2, 3
6.	**r**	Implication Elimination: 4, 5

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 6 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$p \Rightarrow q$	Assumption
3.	p	Assumption
4.	$q \Rightarrow r$	Implication Elimination: 1, 3
5.	q	Implication Elimination: 2, 3
6.	r	Implication Elimination: 4, 5
7.	**$p \Rightarrow r$**	Implication Introduction: 3, 6

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 7 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$p \Rightarrow q$	Assumption
3.	p	Assumption
4.	$q \Rightarrow r$	Implication Elimination: 1, 3
5.	q	Implication Elimination: 2, 3
6.	r	Implication Elimination: 4, 5
7.	$p \Rightarrow r$	Implication Introduction: 3, 6
8.	**$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$**	Implication Introduction: 2, 7

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Step 8 of 9

Fitch proof practice 1

1.	$p \Rightarrow (q \Rightarrow r)$	Assumption
2.	$\quad p \Rightarrow q$	Assumption
3.	$\quad\quad p$	Assumption
4.	$\quad\quad q \Rightarrow r$	Implication Elimination: 1, 3
5.	$\quad\quad q$	Implication Elimination: 2, 3
6.	$\quad\quad r$	Implication Elimination: 4, 5
7.	$\quad p \Rightarrow r$	Implication Introduction: 3, 6
8.	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	Implication Introduction: 2, 7
9.	$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	Implication Introduction: 1, 8

GOAL

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Complete

Fitch proof practice 2: implication as disjunction

Prove the following using Fitch-style natural deduction.

PREMISE

$p \Rightarrow q$

GOAL

$\neg p \vee q$

Fitch proof practice 2: implication as disjunction

1. $p \Rightarrow q$ Premise

PREMISE

$p \Rightarrow q$

GOAL

$\neg p \vee q$

Step 1 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	| ¬(¬p ∨ q)	Assumption

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 2 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	| ¬(¬p ∨ q)	Assumption
3.	| | **p**	Assumption

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 3 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	**q**	Implication Elimination: 1, 3

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 4 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	**¬p ∨ q**	Or Introduction: 4

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 5 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	**p ⇒ ¬p ∨ q**	Implication Introduction: 3, 5

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 6 of 22

Fitch proof practice 2: implication as disjunction

1.	$p \Rightarrow q$	Premise
2.	$\quad \neg(\neg p \vee q)$	Assumption
3.	$\quad\quad p$	Assumption
4.	$\quad\quad q$	Implication Elimination: 1, 3
5.	$\quad\quad \neg p \vee q$	Or Introduction: 4
6.	$\quad p \Rightarrow \neg p \vee q$	Implication Introduction: 3, 5
7.	$\quad\quad \mathbf{p}$	Assumption

PREMISE

$p \Rightarrow q$

GOAL

$\neg p \vee q$

Step 7 of 22

Fitch proof practice 2: implication as disjunction

1.	$p \Rightarrow q$	Premise
2.	$\neg(\neg p \vee q)$	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	$\neg p \vee q$	Or Introduction: 4
6.	$p \Rightarrow \neg p \vee q$	Implication Introduction: 3, 5
7.	p	Assumption
8.	**$\neg(\neg p \vee q)$**	Reiteration: 2

PREMISE

$p \Rightarrow q$

GOAL

$\neg p \vee q$

Step 8 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	**p ⇒ ¬(¬p ∨ q)**	Implication Introduction: 7, 8

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 9 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	**¬p**	Negation Introduction: 6, 9

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 10 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	**¬(¬p ∨ q) ⇒ ¬p**	Implication Introduction: 2, 10

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 11 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	**¬(¬p ∨ q)**	Assumption

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 12 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	**¬p**	Assumption

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 13 of 22

Fitch proof practice 2: implication as disjunction

1.	$p \Rightarrow q$	Premise
2.	$\quad \neg(\neg p \vee q)$	Assumption
3.	$\quad\quad p$	Assumption
4.	$\quad\quad q$	Implication Elimination: 1, 3
5.	$\quad\quad \neg p \vee q$	Or Introduction: 4
6.	$\quad p \Rightarrow \neg p \vee q$	Implication Introduction: 3, 5
7.	$\quad\quad p$	Assumption
8.	$\quad\quad \neg(\neg p \vee q)$	Reiteration: 2
9.	$\quad p \Rightarrow \neg(\neg p \vee q)$	Implication Introduction: 7, 8
10.	$\quad \neg p$	Negation Introduction: 6, 9
11.	$\neg(\neg p \vee q) \Rightarrow \neg p$	Implication Introduction: 2, 10
12.	$\quad \neg(\neg p \vee q)$	Assumption
13.	$\quad\quad \neg p$	Assumption
14.	$\quad\quad \mathbf{\neg p \vee q}$	Or Introduction: 13

PREMISE

$p \Rightarrow q$

GOAL

$\neg p \vee q$

Step 14 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	**¬p ⇒ ¬p ∨ q**	Implication Introduction: 13, 14

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 15 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	¬p ⇒ ¬p ∨ q	Implication Introduction: 13, 14
16.	**¬p**	Assumption

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 16 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	¬p ⇒ ¬p ∨ q	Implication Introduction: 13, 14
16.	¬p	Assumption
17.	**¬(¬p ∨ q)**	Reiteration: 12

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 17 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	¬p ⇒ ¬p ∨ q	Implication Introduction: 13, 14
16.	¬p	Assumption
17.	¬(¬p ∨ q)	Reiteration: 12
18.	**¬p ⇒ ¬(¬p ∨ q)**	Implication Introduction: 16, 17

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 18 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	¬p ⇒ ¬p ∨ q	Implication Introduction: 13, 14
16.	¬p	Assumption
17.	¬(¬p ∨ q)	Reiteration: 12
18.	¬p ⇒ ¬(¬p ∨ q)	Implication Introduction: 16, 17
19.	**¬¬p**	Negation Introduction: 15, 18

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 19 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	¬p ⇒ ¬p ∨ q	Implication Introduction: 13, 14
16.	¬p	Assumption
17.	¬(¬p ∨ q)	Reiteration: 12
18.	¬p ⇒ ¬(¬p ∨ q)	Implication Introduction: 16, 17
19.	¬¬p	Negation Introduction: 15, 18
20.	**¬(¬p ∨ q) ⇒ ¬¬p**	Implication Introduction: 12, 19

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 20 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	¬p ⇒ ¬p ∨ q	Implication Introduction: 13, 14
16.	¬p	Assumption
17.	¬(¬p ∨ q)	Reiteration: 12
18.	¬p ⇒ ¬(¬p ∨ q)	Implication Introduction: 16, 17
19.	¬¬p	Negation Introduction: 15, 18
20.	¬(¬p ∨ q) ⇒ ¬¬p	Implication Introduction: 12, 19
21.	**¬¬(¬p ∨ q)**	Negation Introduction: 11, 20

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Step 21 of 22

Fitch proof practice 2: implication as disjunction

1.	p ⇒ q	Premise
2.	¬(¬p ∨ q)	Assumption
3.	p	Assumption
4.	q	Implication Elimination: 1, 3
5.	¬p ∨ q	Or Introduction: 4
6.	p ⇒ ¬p ∨ q	Implication Introduction: 3, 5
7.	p	Assumption
8.	¬(¬p ∨ q)	Reiteration: 2
9.	p ⇒ ¬(¬p ∨ q)	Implication Introduction: 7, 8
10.	¬p	Negation Introduction: 6, 9
11.	¬(¬p ∨ q) ⇒ ¬p	Implication Introduction: 2, 10
12.	¬(¬p ∨ q)	Assumption
13.	¬p	Assumption
14.	¬p ∨ q	Or Introduction: 13
15.	¬p ⇒ ¬p ∨ q	Implication Introduction: 13, 14
16.	¬p	Assumption
17.	¬(¬p ∨ q)	Reiteration: 12
18.	¬p ⇒ ¬(¬p ∨ q)	Implication Introduction: 16, 17
19.	¬¬p	Negation Introduction: 15, 18
20.	¬(¬p ∨ q) ⇒ ¬¬p	Implication Introduction: 12, 19
21.	¬¬(¬p ∨ q)	Negation Introduction: 11, 20
22.	¬p ∨ q	Negation Elimination: 21

PREMISE

p ⇒ q

GOAL

¬p ∨ q

Complete

Fitch proof practice 3: De Morgan's law

Prove the following using Fitch-style natural deduction.

PREMISE

$\neg(p \vee q)$

GOAL

$\neg p \wedge \neg q$

Fitch proof practice 3: De Morgan's law

1. **¬(p ∨ q)** Premise

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 1 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| **p**	Assumption

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 2 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| **p ∨ q**	Or Introduction: 2

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 3 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	**p ⇒ p ∨ q**	Implication Introduction: 2, 3

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 4 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| **p**	Assumption

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 5 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| **¬(p ∨ q)**	Reiteration: 1

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 6 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	**p ⇒ ¬(p ∨ q)**	Implication Introduction: 5, 6

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 7 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	**¬p**	Negation Introduction: 4, 7

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 8 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| **q**	Assumption

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 9 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| q	Assumption
10.	| **p ∨ q**	Or Introduction: 9

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 10 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| q	Assumption
10.	| p ∨ q	Or Introduction: 9
11.	**q ⇒ p ∨ q**	Implication Introduction: 9, 10

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 11 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| q	Assumption
10.	| p ∨ q	Or Introduction: 9
11.	q ⇒ p ∨ q	Implication Introduction: 9, 10
12.	| **q**	Assumption

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 12 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| q	Assumption
10.	| p ∨ q	Or Introduction: 9
11.	q ⇒ p ∨ q	Implication Introduction: 9, 10
12.	| q	Assumption
13.	| **¬(p ∨ q)**	Reiteration: 1

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 13 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| q	Assumption
10.	| p ∨ q	Or Introduction: 9
11.	q ⇒ p ∨ q	Implication Introduction: 9, 10
12.	| q	Assumption
13.	| ¬(p ∨ q)	Reiteration: 1
14.	**q ⇒ ¬(p ∨ q)**	Implication Introduction: 12, 13

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 14 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| q	Assumption
10.	| p ∨ q	Or Introduction: 9
11.	q ⇒ p ∨ q	Implication Introduction: 9, 10
12.	| q	Assumption
13.	| ¬(p ∨ q)	Reiteration: 1
14.	q ⇒ ¬(p ∨ q)	Implication Introduction: 12, 13
15.	**¬q**	Negation Introduction: 11, 14

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Step 15 of 16

Fitch proof practice 3: De Morgan's law

1.	¬(p ∨ q)	Premise
2.	| p	Assumption
3.	| p ∨ q	Or Introduction: 2
4.	p ⇒ p ∨ q	Implication Introduction: 2, 3
5.	| p	Assumption
6.	| ¬(p ∨ q)	Reiteration: 1
7.	p ⇒ ¬(p ∨ q)	Implication Introduction: 5, 6
8.	¬p	Negation Introduction: 4, 7
9.	| q	Assumption
10.	| p ∨ q	Or Introduction: 9
11.	q ⇒ p ∨ q	Implication Introduction: 9, 10
12.	| q	Assumption
13.	| ¬(p ∨ q)	Reiteration: 1
14.	q ⇒ ¬(p ∨ q)	Implication Introduction: 12, 13
15.	¬q	Negation Introduction: 11, 14
16.	¬p ∧ ¬q	And Introduction: 8, 15

PREMISE

¬(p ∨ q)

GOAL

¬p ∧ ¬q

Complete

06

Resolution

Clausal form, one rule, and proof by refutation

Resolution proofs: clausal form

Literal

An atom or its negation.

$p \quad \neg p$

Clause

A set of literals. Order and duplicates do not matter.

$\{p\} \quad \{\neg p, q\} \quad \{\}$

The empty clause {} is a contradiction (unsatisfiable).

Clausal sentence

A literal or a disjunction of literals.

$p \vee \neg q \vee r$

A set of clauses is a conjunction of those disjunctions.

Clausal form: the conversion pipeline

Do the steps in order: I, N, D, O

I

Implications

Rewrite them away.

$\varphi \Rightarrow \psi$ becomes

$\neg\varphi \vee \psi$

$\varphi \Leftrightarrow \psi$ becomes

$(\neg\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$

›

N

Negations

Push $\neg$ inward.

De Morgan's laws

Double negation

›

D

Distribution

Distribute $\vee$ over $\wedge$.

Result: a conjunction of disjunctions

›

O

Operators

Split into sets of literals.

Each disjunction becomes one clause

The conjunction becomes a set of clauses

The resolution rule

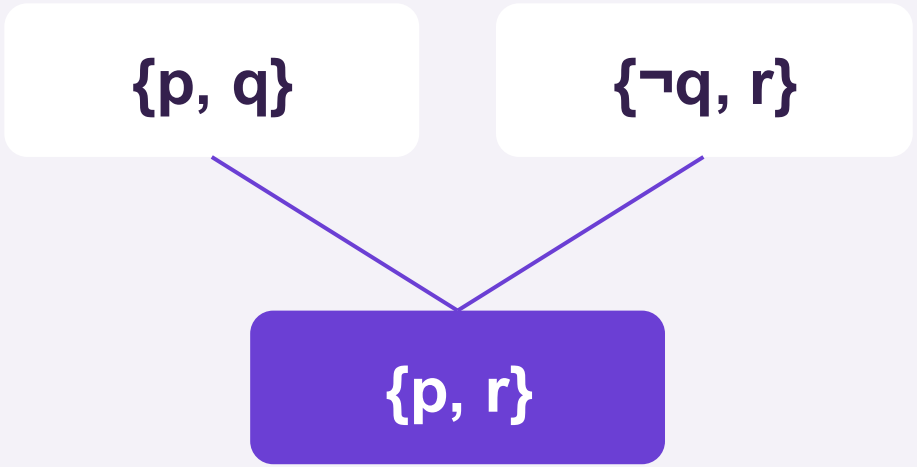
If two clauses contain complementary literals χ and $\neg\chi$, you can resolve them.

$\{\ldots, \chi, \ldots\}$ and $\{\ldots, \neg\chi, \ldots\}$

give {all the other literals combined}

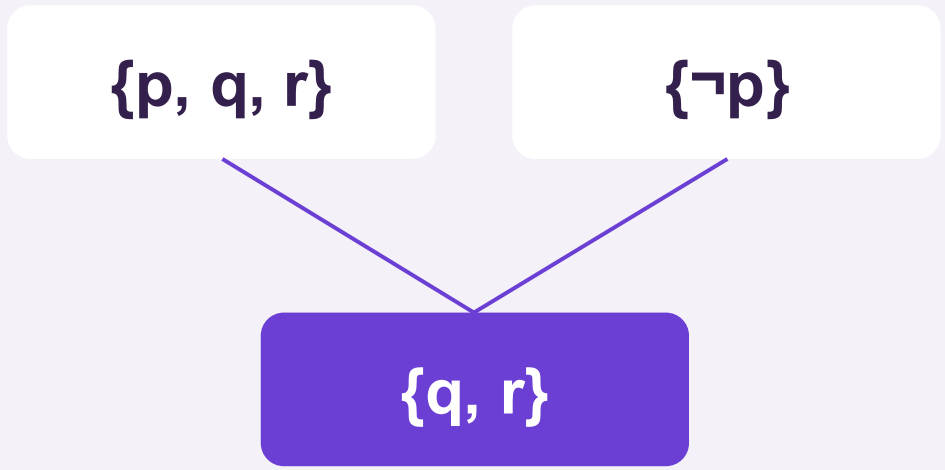
Note: cancel only one complementary pair at a time.

Example

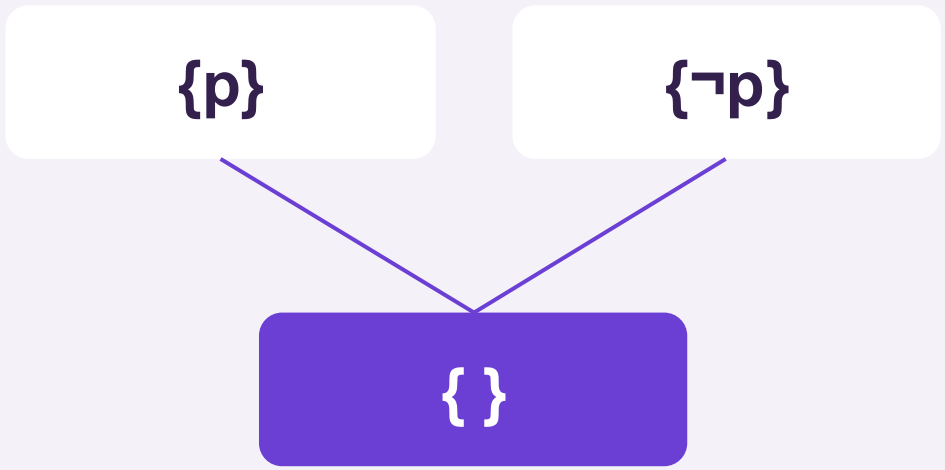


q and ¬q cancel; everything else is kept.

The resolution rule: singleton effects



Resolving with a single literal removes that literal from the other clause.



The empty clause means a contradiction has been found.

Derivations vs. proofs

Resolution derivation

Start from the premise clauses and repeatedly resolve to reach a target clause.

Not complete for generating all entailed clauses.

Resolution proof

Refutation style. To prove $\Delta \vDash \varphi$:

1. Take clauses(Δ) and clauses($\neg\varphi$).
2. Resolve until you derive $\{\}$.

Complete for unsatisfiability (refutation-complete).

Why resolution is nice

1

Focused search

No hunt for axiom instances. Just pair clauses that have complementary literals.

2

Termination

With finitely many symbols, only finitely many distinct clauses exist, so the search can saturate and stop.

Helpful patterns

Map implications to clauses

$p \Rightarrow q$ becomes $\{\neg p, q\}$. With $\{p\}$ it resolves to $\{q\}$, which is Modus Ponens.

Clauses behave like sets

Drop duplicates in resolvents. The order of literals is irrelevant.

Do not over-resolve

If both p and q are complementary across two clauses, you may cancel only one pair per step.

Reaching the empty clause

An inconsistency has been found, so the goal is proved by refutation.

How to use resolution

1

Step 1

Convert

Convert the premises and the negated goal to clauses, using I, N, D, O.

2

Step 2

Resolve

Resolve pairs of clauses with complementary literals, and add the new clauses.

3

Step 3

Stop

If you derive { }, the goal is proved. If no new clauses appear (saturation) and there is no empty clause, it is not proved by refutation.

Resolution proof example

1.	{¬p, q}	Premise
2.	{¬r, s}	Premise
3.	{p, r}	Negated goal
4.	{¬q}	Negated goal
5.	**{¬s}**	Negated goal

METHOD

Resolution refutation

GOAL

{ } (the empty clause)

Step 5 of 9

Resolution proof example

1.	{¬p, q}	Premise
2.	{¬r, s}	Premise
3.	{p, r}	Negated goal
4.	{¬q}	Negated goal
5.	{¬s}	Negated goal
6.	**{¬p}**	Resolution: 1, 4

METHOD

Resolution refutation

GOAL

{ } (the empty clause)

Step 6 of 9

Resolution proof example

1.	{¬p, q}	Premise
2.	{¬r, s}	Premise
3.	{p, r}	Negated goal
4.	{¬q}	Negated goal
5.	{¬s}	Negated goal
6.	{¬p}	Resolution: 1, 4
7.	**{¬r}**	Resolution: 2, 5

METHOD

Resolution refutation

GOAL

{ } (the empty clause)

Step 7 of 9

Resolution proof example

1.	{¬p, q}	Premise
2.	{¬r, s}	Premise
3.	{p, r}	Negated goal
4.	{¬q}	Negated goal
5.	{¬s}	Negated goal
6.	{¬p}	Resolution: 1, 4
7.	{¬r}	Resolution: 2, 5
8.	**{r}**	Resolution: 3, 6

METHOD

Resolution refutation

GOAL

{ } (the empty clause)

Step 8 of 9

Resolution proof example

1.	{¬p, q}	Premise
2.	{¬r, s}	Premise
3.	{p, r}	Negated goal
4.	{¬q}	Negated goal
5.	{¬s}	Negated goal
6.	{¬p}	Resolution: 1, 4
7.	{¬r}	Resolution: 2, 5
8.	{r}	Resolution: 3, 6
9.	{ }	Resolution: 8, 7

METHOD

Resolution refutation

GOAL

{ } (the empty clause)

Proved

Thank you

Good luck on Quiz 1

