# Quiz 2 Review

# Relational Logic

# Relational Logic

Object constants: a, b

Relational constants: p, q, r

Variables: x, y, z

p and q are 1-ary, r is 2-ary

Ground terms: a,b

Ground atomic sentences:

p(a), p(b), q(a), q(b), r(a,a), r(a,b), r(b,a), r(b,b)

Each ground atomic sentence has a truth assignment, and the model is defined by these truth assignments

# Relational Logic: Evaluations

p(a) TRUE

p(b) FALSE

q(a,a) TRUE

q(a,b) TRUE

q(b,a) TRUE

q(b,b) FALSE

- $q(a, b) \to p(a)$
- $\exists x.p(x)$
- $\forall x.q(x, x)$
- $\forall x.\exists y.q(x, y)$
- $\exists x.\forall y.\neg q(x, y)$
- $\forall y.(\exists x.\neg q(x, y) \to \neg p(y))$
- $\exists y.\forall z.(p(z) \leftrightarrow q(y, z))$
- $\exists x.\forall y.q(x, y) \leftrightarrow \forall x.\exists y.q(x, y)$

# Valid, contingent or unsatisfiable

$\exists x.\forall y.p(x,y) \rightarrow \forall y.\exists x.p(x,y)$

$\forall y.\exists x.p(x,y) \rightarrow \exists x.\forall y.p(x,y)$

$\forall x.(p(x) \rightarrow q(x)) \rightarrow (\neg q(a) \rightarrow \neg p(a))$

$\forall x.(p(x) \rightarrow q(x)) \rightarrow (q(a) \rightarrow p(a))$

$(\forall x.\forall y.p(x,y) \rightarrow \forall z.p(z,z)) \rightarrow \forall x.(p(x) \wedge \neg p(x))$

$\exists x.p(x) \wedge \forall x.\neg p(x)$

$\exists x.p(x) \vee \forall x.\neg p(x)$

$\forall x.\forall y.p(x,y) \leftrightarrow \forall z.p(z,z)$

# True/False

If $\Gamma \models p(a) \vee q(a)$ and $\Delta \models \forall x.\neg p(x)$, then $\Gamma \cup \Delta \models q(a)$

If $\Gamma \models p(a) \vee q(a)$ and $\Delta \models \forall x.\neg p(x)$, then $\Gamma \cap \Delta \models q(a)$

If $\Gamma \models \phi$ and $\phi \models \psi$, then $\Gamma \models \psi$.

If $\Gamma \models \phi$ and $\Gamma \models \psi$, then $\phi \models \psi$.

If $\Gamma \models p(t)$ for some ground term $t$, then $\Gamma \models \exists x.p(x)$.

If $\Gamma \models p(t)$ for every ground term $t$, then $\Gamma \models \forall x.p(x)$.

If $\Gamma \models \forall x.p(x)$, then $\Gamma \models p(t)$ for every ground term $t$.

If $\Gamma \models \exists x.p(x)$, then $\Gamma \models p(t)$ for some ground term $t$.

# Logic Board

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **1** | | | | |
| **2** | | | | |
| **3** | | | | |
| **4** | | | | |

$gold(2,3) \vee gold(2,4)$

$\neg gold(1,4) \wedge \neg gold(4,1)$

$\forall y.(gold(2,y) \Rightarrow gold(1,y))$

$\forall x.(gold(x,3) \Rightarrow gold(3,x))$

$\exists y.(gold(2,y) \wedge gold(y,2))$

$\neg \exists x.gold(x,x)$

$\forall x.\exists y.gold(x,y)$

# Sentences:

- $\forall x.(\neg likes(x,3) \rightarrow likes(3,x))$
- $\neg(likes(1,2) \wedge likes(3,4))$
- $\forall x.\neg likes(4,x)$
- $\forall x.\exists y.\neg likes(x,y)$
- $\neg(likes(1,3))$
- $likes(1,1) \leftrightarrow (likes(2,4) \wedge likes(3,2))$

# Fitch Proofs

# New Rules of Inference

**Universal Introduction**

$$\frac{\varphi}{\forall \nu.\varphi_{\tau \Leftarrow \nu}}$$

where τ is not used in any active assumptions

**Universal Elimination**

$$\frac{\forall \nu.\varphi}{\varphi_{\nu \Leftarrow \tau}}$$

where τ is ground

**Existential Introduction**

$$\frac{\varphi}{\exists \nu.\varphi_{\tau \leftarrow \nu}}$$

where τ is ground

**Existential Elimination**

$$\frac{\exists \nu.\varphi \qquad \forall \nu.(\varphi \Rightarrow \psi)}{\psi}$$

where ν does not occur free in ψ

# Practice Proof 1

Premises: AX:AY:(p(X,Y) | p(Y,X)), ~p(a,b)

Goal: p(b,a)

| | | |
|---|---|---|
| ☐ 1. | AX:AY:(p(X,Y) \| p(Y,X)) | Premise |
| ☐ 2. | ~p(a,b) | Premise |
| ☐ 3. | AY:(p(a,Y) \| p(Y,a)) | Universal Elimination: 1 |
| ☐ 4. | p(a,b) \| p(b,a) | Universal Elimination: 3 |
| ☐ 5. | p(a,b) | Assumption |
| ☐ 6. | ~p(b,a) | Assumption |
| ☐ 7. | p(a,b) | Reiteration: 5 |
| ☐ 8. | ~p(b,a) => p(a,b) | Implication Introduction: 6, 7 |
| ☐ 9. | ~p(b,a) | Assumption |
| ☐ 10. | ~p(a,b) | Reiteration: 2 |
| ☐ 11. | ~p(b,a) => ~p(a,b) | Implication Introduction: 9, 10 |
| ☐ 12. | ~~p(b,a) | Negation Introduction: 8, 11 |
| ☐ 13. | p(b,a) | Negation Elimination: 12 |
| ☐ 14. | p(a,b) => p(b,a) | Implication Introduction: 5, 13 |
| ☐ 15. | p(b,a) | Assumption |
| ☐ 16. | p(b,a) => p(b,a) | Implication Introduction: 15, 15 |
| ☐ 17. | p(b,a) | Or Elimination: 4, 14, 16 |

# Practice Proof 2

Premises: AX:AY:(p(X,Y) | p(Y,X)), AX:AY:~p(X,Y)

Goal: AX:q(X)

| | | |
|---|---|---|
| ☐ 1. | AX:AY:(p(X,Y) \| p(Y,X)) | Premise |
| ☐ 2. | AX:AY:~p(X,Y) | Premise |
| ☐ 3. | AY:~p(a,Y) | Universal Elimination: 2 |
| ☐ 4. | AY:(p(a,Y) \| p(Y,a)) | Universal Elimination: 1 |
| ☐ 5. | ~p(a,a) | Universal Elimination: 3 |
| ☐ 6. | p(a,a) \| p(a,a) | Universal Elimination: 4 |
| ☐ 7. | p(a,a) | Assumption |
| ☐ 8. | ~q(b) | Assumption |
| ☐ 9. | p(a,a) | Reiteration: 7 |
| ☐ 10. | ~q(b) => p(a,a) | Implication Introduction: 8, 9 |
| ☐ 11. | ~q(b) | Assumption |
| ☐ 12. | ~p(a,a) | Reiteration: 5 |
| ☐ 13. | ~q(b) => ~p(a,a) | Implication Introduction: 11, 12 |
| ☐ 14. | ~~q(b) | Negation Introduction: 10, 13 |
| ☐ 15. | q(b) | Negation Elimination: 14 |
| ☐ 16. | p(a,a) => q(b) | Implication Introduction: 7, 15 |
| ☐ 17. | q(b) | Or Elimination: 6, 16, 16 |
| ☐ 18. | AX:q(X) | Universal Introduction: 17 |

# Existential Elimination (the hard one!)

You can think of EE as Or Elimination on steroids!

Five steps to using existential elimination:

1. Assume the scope of the existential
2. Prove your goal under the assumption
3. Exit the assumption with Implication Introduction
4. Use Universal Introduction on the free variable in the antecedent
5. Use Existential Elimination!

# Practice Proof 3

Premise: EX:(p(X) & q(X))
Goal: EX:p(X)

| | | |
|---|---|---|
| 1. | EX:(p(X) & q(X)) | Premise |
| 2. | p(a) & q(a) | Assumption |
| 3. | p(a) | And Elimination: 2 |
| 4. | EX:p(X) | Existential Introduction: 3 |
| 5. | p(a) & q(a) => EX:p(X) | Implication Introduction: 2, 4 |
| 6. | AX:(p(X) & q(X) => EX:p(X)) | Universal Introduction: 5 |
| 7. | EX:p(X) | Existential Elimination: 1, 6 |

# Practice Proof 4

Premises: AX:AY:(p(X,Y) ⇔ q(Y,Y)), EX:q(X,X)
Goal: AX:EY:p(X,Y)

| | | |
|---|---|---|
| ☐ 1. | EX:q(X,X) | Premise |
| ☐ 2. | AX:AY:(p(X,Y) <=> q(Y,Y)) | Premise |
| ☐ 3. | AY:(p(a,Y) <=> q(Y,Y)) | Universal Elimination: 2 |
| ☐ 4. | p(a,b) <=> q(b,b) | Universal Elimination: 3 |
| ☐ 5. | q(b,b) => p(a,b) | Biconditional Elimination: 4 |
| ☐ 6. | │ q(b,b) | Assumption |
| ☐ 7. | │ p(a,b) | Implication Elimination: 5, 6 |
| ☐ 8. | │ EY:p(a,Y) | Existential Introduction: 7 |
| ☐ 9. | q(b,b) => EY:p(a,Y) | Implication Introduction: 6, 8 |
| ☐ 10. | AX:(q(X,X) => EY:p(a,Y)) | Universal Introduction: 9 |
| ☐ 11. | EY:p(a,Y) | Existential Elimination: 1, 10 |
| ☐ 12. | AX:EY:p(X,Y) | Universal Introduction: 11 |

# Q&A