Agenda

1. Validity, Unsatisfiability, Contingency
2. Logical Entailment, Equivalence, Consistency
3. Soundness and Completeness
4. Hilbert Proofs
5. Natural Deduction & Fitch Proofs
6. Resolution
Validity, Unsatisfiability, Contingency
Propositional Logic & Analysis

→ **Valid**: Satisfied by *every truth assignment*.
  - E.g. \((p \lor \neg p)\).
  - Always yields T in a truth table

→ **Unsatisfiable**: Not satisfied by *any* truth assignment.
  - E.g. \((p \land \neg p)\)
  - Always yields F in a truth table

→ **Contingent**: There is some truth assignment that satisfies it, and some truth assignment that falsifies it
  - E.g. \((p \land q)\): If \(p\) and \(q\) both true, it is true. If \(p\) and \(q\) both false, it is false
Logical Entailment, Equivalence, Consistency
Logical Entailment

- A set of sentences logically entails a sentence iff every truth assignment that satisfies the premises also satisfies the conclusion

- Example 1:
  - \( \{p \Rightarrow r\} \vdash (p \Rightarrow q \lor r) \)

- Example 2:
  - \( \{p \Rightarrow q \lor r, p \Rightarrow r\} \vdash (q \Rightarrow r) \)

- Extra Practice:
  - [http://intrologic.stanford.edu/exercises/exercise_03_03.html](http://intrologic.stanford.edu/exercises/exercise_03_03.html)
  - [http://intrologic.stanford.edu/exercises/exercise_03_04.html](http://intrologic.stanford.edu/exercises/exercise_03_04.html)
Exercise 3.4.3

If $\Gamma \models \phi$ and $\Delta \not\models \phi$, then $\Gamma \cup \Delta \models \phi$
For the second question $\Gamma \models \varphi$ and $\Delta \not\models \varphi$, then $\Gamma \cup \Delta \models \varphi$. WLOG, this time let $\Gamma = \{a, b\}$ and $\Delta = \{b, c\}$. $\Gamma \models \varphi$ means any truth assignment that satisfies $a$ and $b$ also satisfy $\varphi$, i.e. $a \land b \Rightarrow \varphi$. $\Delta \not\models \varphi$ means some truth assignment that satisfies $b$ and $c$ does not satisfy $\varphi$. Now $\Gamma \cup \Delta = \{a, b, c\}$, so the question is about whether $a \land b \land c \Rightarrow \varphi$. Notice that any truth assignment that satisfies $a \land b \land c$ must satisfy $a \land b$ as well, so it also satisfies $\varphi$, and we have that $\Gamma \cup \Delta$ entails $\varphi$. The fact that $\Delta \not\models \varphi$ has nothing to do with it, since we know $\varphi$ must be true as long as $a \land b$ is true.

IMO, you just need to be aware that $\Gamma \cup \Delta \not= \Gamma \lor \Delta$ - this seems to be an easy mistake.
Logical Equivalence

- \( \phi \) and \( \Psi \) are logically equivalent if they entail each other
- Equivalence Theorem:
  - \( \phi \) and \( \Psi \) are equivalent if \( (\phi \iff \Psi) \)
- Example 1:
  - \( ((p \Rightarrow q) \lor (q \Rightarrow r)) \) and \( (p \lor \neg p) \)
- Example 2:
  - \( (p \land q \Rightarrow r) \) and \( (p \land r \Rightarrow q) \)
- Extra Practice: [http://intrologic.stanford.edu/exercises/exercise_03_02.html](http://intrologic.stanford.edu/exercises/exercise_03_02.html)
Logical Consistency

- \( \phi \) and \( \Psi \) are consistent if there is a truth assignment that satisfies both.

- Example 1:
  - \( \{p \Rightarrow r, q \Rightarrow r, p \lor q\} \) and \( r \)

- Example 2:
  - \( \{p \Rightarrow r, q \Rightarrow r, p \lor q\} \) and \( \neg r \)

- Extra Practice: [http://intrologic.stanford.edu/exercises/exercise_03_05.html](http://intrologic.stanford.edu/exercises/exercise_03_05.html)
Soundness & Completeness
Soundness and Completeness

- **Soundness**: a proof system is sound iff every conclusion that is provable from a set of premises is logically entailed.
  - $\Delta \vdash \phi$, then $\Delta \models \phi$
  - Everything derivable / provable is true

- **Completeness**: a proof system is complete iff every conclusion that is logically entailed by a set of premises is provable
  - $\Delta \models \phi$, then $\Delta \vdash \phi$
  - Everything true is derivable / provable

Hilbert? Yes and Yes
Fitch? Yes and Yes
Hilbert Proofs
Hilbert Proofs

Implication Elimination

\[ \phi \Rightarrow \psi \\
\phi \\
\hline
\psi \]

Implication Creation (IC)  \[ \phi \Rightarrow (\psi \Rightarrow \phi) \]

Implication Distribution (ID)  \[ (\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi)) \]

Implication Reversal (IR)  \[ (\neg \psi \Rightarrow \neg \phi) \Rightarrow (\phi \Rightarrow \psi) \]
Live Demo. Premises: \( p \) and \( \sim p \) Prove: \( q \)

http://intrologic.stanford.edu/logica/
hompage/hilbert.php
Introduction to Logic

Hilbert

<table>
<thead>
<tr>
<th></th>
<th>Undo</th>
<th>Copy</th>
<th>Paste</th>
<th>Load</th>
<th>Save</th>
<th>Help</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>p</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>~p</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>/~q =&gt; ~p</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>~q =&gt; ~p</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>(~q =&gt; ~p) =&gt; (p =&gt; q)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>p =&gt; q</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>q</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Goal | q Complete

<table>
<thead>
<tr>
<th></th>
<th>Premise</th>
<th>Implication Creation</th>
<th>Implication Elimination</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reiteration</td>
<td>Implication Distribution</td>
<td>Implication Reversal</td>
<td>Universal Generalization</td>
</tr>
<tr>
<td>Truthtable</td>
<td>Shortcut</td>
<td>Universal Specialization</td>
<td>Domain Closure</td>
</tr>
<tr>
<td>Replace</td>
<td>Coalesce</td>
<td>Universal Distribution</td>
<td>Induction</td>
</tr>
</tbody>
</table>

Show Answer  Reset
Live Demo. Premises: ~q and ~p => (~q=>~r) Prove: r=>p

http://intrologic.stanford.edu/logica/
homepage/hilbert.php
Practice Test - Problem 3

1. \neg q
2. \neg p \implies (\neg q \implies \neg r)
3. r \implies p
4. (\neg p \implies (\neg q \implies \neg r)) \implies ((\neg p \implies \neg q) \implies (\neg p \implies \neg r))
5. (\neg p \implies \neg q) \implies (\neg p \implies \neg r)
6. \neg q \implies (\neg p \implies \neg q)
7. \neg p \implies \neg q
8. \neg p \implies \neg r
9. (\neg p \implies \neg r) \implies (r \implies p)
10. r \implies p

Goal: p

Premise
Premise
Goal
Implication Distribution
Implication Elimination: 4, 2
Implication Creation
Implication Elimination: 6, 1
Implication Elimination: 5, 7
Implication Reversal
Implication Elimination: 9, 8

Incomplete

<table>
<thead>
<tr>
<th>Goal</th>
<th>Premise</th>
<th>Implication Creation</th>
<th>Implication Elimination</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retraction</td>
<td>Implication Distribution</td>
<td>Universal Generalization</td>
<td></td>
</tr>
<tr>
<td>Truth Table</td>
<td>Implication Reversal</td>
<td>Domain Closure</td>
<td></td>
</tr>
<tr>
<td>Shortcut</td>
<td>Universal Specialization</td>
<td>Induction</td>
<td></td>
</tr>
<tr>
<td>Replace</td>
<td>Universal Distribution</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Dominic’s Qn

1. $p \iff \neg p$
   - Premise
   - Biconditional Elimination: 1

2. $p \Rightarrow \neg p$
   - Assumption
   - Implication Elimination: 2, 4

3. $\neg p \Rightarrow p$
   - Implication Elimination: 3, 5
   - Implication Introduction: 4, 6

4. $p$

5. $\neg p$

6. $p$

7. $p \Rightarrow p$
   - Implication Introduction: 7, 2

8. $\neg p$
   - Implication Elimination: 3, 9
   - Implication Elimination: 2, 10

9. $\neg p$

10. $p$

11. $\neg p$

12. $\neg p \Rightarrow \neg p$

13. $\neg \neg p$
   - Negation Introduction: 13

14. $p$

15. $\neg q$
   - Assumption
   - Reiteration: 14

16. $p$

17. $\neg q \Rightarrow p$
   - Implication Introduction: 15, 16

18. $\neg q$

19. $\neg p$

20. $\neg q \Rightarrow \neg p$

21. $\neg \neg q$
   - Negation Introduction: 17, 20

22. $q$
   - Negation Elimination: 21
Natural Deduction & Fitch Proofs
Natural Deduction: Fitch Proofs

Several comments on Ed / in OH about how Fitch Proofs seem unintuitive / unnatural (especially when making assumptions)
- This is normal, and you will get better with practice!
- Resources: Lesson Exercises (especially 5.11-5.14, some of which we will go through), Practice Quiz, Past Quizzes

Some things to note
- Don’t just focus on the premises (what you have to work with), take note of the goal (where you’re trying to go)
  - This will help with 1. choice of assumptions / subproofs and 2. choice of rules
- Be familiar with your toolbox i.e. the rules you can use
- Mindset: given the info that I have (premises) and the tools I have to work with (rules), how do I get to the goal
Natural Deduction: Fitch Proofs

And Introduction
\[ \phi_1 \]
\[ \ldots \]
\[ \phi_n \]
\[ \phi_1 \land \ldots \land \phi_n \]

And Elimination
\[ \phi_1 \land \ldots \land \phi_n \]
\[ \phi_i \]

Or Introduction
\[ \phi_i \]
\[ \phi_1 \lor \ldots \lor \phi_n \]

Or Elimination
\[ \phi_1 \lor \ldots \lor \phi_n \]
\[ \phi_1 \rightarrow \psi \]
\[ \ldots \]
\[ \phi_n \rightarrow \psi \]
\[ \psi \]
Natural Deduction: Fitch Proofs

**Negation Introduction**

\[ \phi \Rightarrow \psi \]
\[ \phi \Rightarrow \neg \psi \]
\[ \neg \phi \]

**Negation Elimination**

\[ \neg \neg \phi \]
\[ \phi \]

**Implication Introduction**

\[ \phi \vdash \psi \]
\[ \phi \Rightarrow \psi \]

**Implication Elimination**

\[ \phi \Rightarrow \psi \]
\[ \phi \]
\[ \psi \]
Biconditional Introduction

\[ \phi \implies \psi \]
\[ \psi \implies \phi \]
\[ \phi \iff \psi \]

Biconditional Elimination

\[ \phi \iff \psi \]
\[ \phi \implies \psi \]
\[ \psi \implies \phi \]
Other things to note about the interface

→ Premise operation: Allows one to add a new premise to a proof

→ Reiteration operation: Allows one to reproduce an earlier conclusion for the purposes of clarity.

→ Delete operation: Allows one to delete unnecessary lines.

→ **Reasoning Tips**: Lecture 5 (Natural Deduction) Slides 39-42
Live Demo

Fitch
Exercise 5.11
Ed Post
### Problem 4 - Fitch

<table>
<thead>
<tr>
<th>No.</th>
<th>Line</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\neg q)</td>
<td>Premise</td>
</tr>
<tr>
<td>2</td>
<td>(\neg p \implies (\neg q \implies \neg r))</td>
<td>Premise</td>
</tr>
<tr>
<td>3</td>
<td>(s \land r)</td>
<td>Premise</td>
</tr>
<tr>
<td>4</td>
<td>(s \implies t)</td>
<td>Premise</td>
</tr>
<tr>
<td>5</td>
<td>(p \implies t)</td>
<td>Premise</td>
</tr>
<tr>
<td>6</td>
<td>(r)</td>
<td>Assumption</td>
</tr>
<tr>
<td>7</td>
<td>(\neg p)</td>
<td>Assumption</td>
</tr>
<tr>
<td>8</td>
<td>(r)</td>
<td>Reiteration: 6</td>
</tr>
<tr>
<td>9</td>
<td>(\neg p \implies r)</td>
<td>Implication Introduction: 7, 8</td>
</tr>
<tr>
<td>10</td>
<td>(\neg p)</td>
<td>Assumption</td>
</tr>
<tr>
<td>11</td>
<td>(\neg q \implies \neg r)</td>
<td>Implication Elimination: 2, 10</td>
</tr>
<tr>
<td>12</td>
<td>(\neg r)</td>
<td>Implication Elimination: 11, 1</td>
</tr>
<tr>
<td>13</td>
<td>(\neg p \implies \neg r)</td>
<td>Implication Introduction: 10, 12</td>
</tr>
<tr>
<td>14</td>
<td>(\neg p)</td>
<td>Negation Introduction: 9, 13</td>
</tr>
<tr>
<td>15</td>
<td>(p)</td>
<td>Negation Elimination: 14</td>
</tr>
<tr>
<td>16</td>
<td>(t)</td>
<td>Implication Elimination: 5, 15</td>
</tr>
<tr>
<td>17</td>
<td>(r \implies t)</td>
<td>Implication Introduction: 6, 16</td>
</tr>
<tr>
<td>18</td>
<td>(t)</td>
<td>Or Elimination: 3, 4, 17</td>
</tr>
</tbody>
</table>

**Goal**: \(t\) Complete
Resolution Proofs
Clausal Form - INDO (I)

Implications:

\[ \phi \Rightarrow \psi \quad \rightarrow \quad \neg \phi \lor \psi \]

\[ \phi \leftrightarrow \psi \quad \rightarrow \quad \phi \lor \neg \psi \]

\[ \phi \leftrightarrow \psi \quad \rightarrow \quad (\neg \phi \lor \psi) \land (\phi \lor \neg \psi) \]
Clausal Form - INDO (N)

Negation:

\[\neg \neg \phi \rightarrow \phi\]
\[\neg (\phi \land \psi) \rightarrow \neg \phi \lor \neg \psi\]
\[\neg (\phi \lor \psi) \rightarrow \neg \phi \land \neg \psi\]
Clausal Form - INDO (I)

Distribution:

\[
\begin{align*}
\phi \lor (\psi \land \chi) & \rightarrow (\phi \lor \psi) \land (\phi \lor \chi) \\
(\phi \land \psi) \lor \chi & \rightarrow (\phi \lor \chi) \land (\psi \lor \chi) \\
\phi \lor (\phi_1 \lor \ldots \lor \phi_n) & \rightarrow \phi \lor \phi_1 \lor \ldots \lor \phi_n \\
(\phi_1 \lor \ldots \lor \phi_n) \lor \phi & \rightarrow \phi_1 \lor \ldots \lor \phi_n \lor \phi \\
\phi \land (\phi_1 \land \ldots \land \phi_n) & \rightarrow \phi \land \phi_1 \land \ldots \land \phi_n \\
(\phi_1 \land \ldots \land \phi_n) \land \phi & \rightarrow \phi_1 \land \ldots \land \phi_n \land \phi
\end{align*}
\]
Clausal Form - INDO (I)

Operators:

\[ \phi_1 \lor \ldots \lor \phi_n \rightarrow \{\phi_1, \ldots, \phi_n\} \]

\[ \phi_1 \land \ldots \land \phi_n \rightarrow \{\phi_1\}, \ldots, \{\phi_n\} \]
Clausal Form Example

\neg (g \land (r \Rightarrow f))
Resolution Principle

\[
\begin{align*}
\{\phi_1, \ldots, \chi, \ldots, \phi_m\} \\
\{\psi_1, \ldots, \neg\chi, \ldots, \psi_n\} \\
\hline
\{\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n\}
\end{align*}
\]

\[
\begin{align*}
\{\neg p, q\} & \quad \{p, q\} & \quad \{p\} & \quad \{p, q\} \\
\{p, q\} & \quad \{\neg q, r\} & \quad \{\neg p\} & \quad \{\neg p, \neg q\} \\
\{q\} & \quad \{p, r\} & \quad \{} & \quad \{p, \neg p\} \\
\{} & \quad \{q, \neg q\} & &
\end{align*}
\]
Resolution Proof

http://intrologic.stanford.edu/exercises/exercise_06_04.html
Given the premises \((p \Rightarrow q)\) and \((r \Rightarrow s)\), use Propositional Resolution to prove the conclusion \((p \lor r \Rightarrow q \lor s)\).

Show Instructions

Select All

<table>
<thead>
<tr>
<th>Resolution</th>
<th></th>
<th>Complete</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\neg p \land q)</td>
<td>Premise</td>
<td></td>
</tr>
<tr>
<td>(\neg r \land s)</td>
<td>Premise</td>
<td></td>
</tr>
<tr>
<td>(p \land r)</td>
<td>Goal</td>
<td></td>
</tr>
<tr>
<td>(\neg q)</td>
<td>Goal</td>
<td></td>
</tr>
<tr>
<td>(\neg s)</td>
<td>Goal</td>
<td></td>
</tr>
<tr>
<td>(\neg p)</td>
<td>Resolution: 1, 4</td>
<td></td>
</tr>
<tr>
<td>(\neg r)</td>
<td>Resolution: 2, 5</td>
<td></td>
</tr>
<tr>
<td>(r)</td>
<td>Resolution: 3, 6</td>
<td></td>
</tr>
<tr>
<td>({})</td>
<td>Resolution: 8, 7</td>
<td></td>
</tr>
</tbody>
</table>
Resolution Proof

### Practice Test - Problem 5

**NB:** We will save and grade only the first 30 lines of your proof. Be economical.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>{p, \neg r}</td>
<td>Premise</td>
</tr>
<tr>
<td>2.</td>
<td>{\neg p, s}</td>
<td>Premise</td>
</tr>
<tr>
<td>3.</td>
<td>{\neg q, \neg r}</td>
<td>Premise</td>
</tr>
<tr>
<td>4.</td>
<td>{q, r}</td>
<td>Premise</td>
</tr>
<tr>
<td>5.</td>
<td>{\neg q, \neg s}</td>
<td>Premise</td>
</tr>
<tr>
<td>6.</td>
<td>{s, p}</td>
<td>Premise</td>
</tr>
<tr>
<td>7.</td>
<td>{\neg p, \neg r, q}</td>
<td>Premise</td>
</tr>
<tr>
<td>8.</td>
<td>{p, \neg q}</td>
<td>Resolution: 6, 5</td>
</tr>
<tr>
<td>9.</td>
<td>{\neg q, x}</td>
<td>Resolution: 8, 2</td>
</tr>
<tr>
<td>10.</td>
<td>{r}</td>
<td>Resolution: 4, 9</td>
</tr>
<tr>
<td>11.</td>
<td>{p}</td>
<td>Resolution: 10, 1</td>
</tr>
<tr>
<td>12.</td>
<td>{\neg q}</td>
<td>Resolution: 10, 3</td>
</tr>
<tr>
<td>13.</td>
<td>{\neg p, \neg r}</td>
<td>Resolution: 7, 12</td>
</tr>
<tr>
<td>14.</td>
<td>{\neg p}</td>
<td>Resolution: 10, 13</td>
</tr>
<tr>
<td>15.</td>
<td>{}</td>
<td>Resolution: 11, 14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Goal</th>
<th>{}</th>
</tr>
</thead>
</table>
Q&A