• Please read all instructions (including these) carefully.

• The exam is closed book, closed notes, and closed internet.

• The exam consists of 11 pages including this page. The last 2 pages are reference sheets for the Fitch rules of inference. There are 5 questions. Each question is worth 10 points.

• Time limit: 2 hours. Budget your time accordingly.

• Please write your solutions in the spaces provided on the exam. Make sure that your solutions are neat and clearly marked. You may use the blank areas and backs of the exam pages for scratch work.

In accordance with both the letter and spirit of the Honor Code, I have neither given nor received assistance on this examination.

NAME: 

SUNETID (username): 

SIGNATURE: 

<table>
<thead>
<tr>
<th>Question 1 (10 points)</th>
<th>Question 2 (10 points)</th>
<th>Question 3 (10 points)</th>
<th>Question 4 (10 points)</th>
<th>Question 5 (10 points)</th>
<th>Total Score (50 points)</th>
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</table>
Question 1 [10 points]

Please circle exactly one of the provided answers (no explanation is required).

(a) $\forall x . (\exists x. p(x) \rightarrow q(x))$

- VALID
- CONTINGENT
- UNSATISFIABLE

(b) $\exists y . (\forall x . (p(x) \land \exists x. q(x)) \rightarrow (\forall y. p(y) \land q(y)))$

- VALID
- CONTINGENT
- UNSATISFIABLE

(c) $(\forall x . p(x) \rightarrow \exists x. p(x)) \rightarrow (\forall x . (q(x) \land p(x)) \land \exists x . \neg(q(x) \lor p(x)))$

- VALID
- CONTINGENT
- UNSATISFIABLE

(d) $\forall x . (\exists x. p(x) \rightarrow (q(x) \land \neg q(x)))$

- VALID
- CONTINGENT
- UNSATISFIABLE
Question 2 [10 points]

For each problem, given the premises, use the Fitch system to prove the desired goal.

Please annotate your proof by writing the rule and line number for each step in your proof (abbreviations are fine). Clearly mark any assumptions and subproofs within your proof in the same format as the exercises and notes.

(a) Given the premise $\exists x. \exists y. (p(x) \lor p(y))$, show the goal $\forall x. \exists y. (p(x) \lor p(y))$.

We have a proof that takes 14 lines.

1. $\exists x. \exists y. (p(x) \lor p(y))$  
   Premise
Question 3 [10 points]

Suppose that our vocabulary consists of the object constants a and b and the binary relation constants p and r. The relation p is axiomatized using the following set of sentences $\Delta$:

\[
\neg p(a, a) \quad \neg p(a, b) \\
p(b, a) \quad p(b, b)
\]

b. Now consider the following sentence $\psi$:

\[
\forall x. \forall y. (p(x, y) \Rightarrow r(y, y))
\]

State (by placing a checkmark $\checkmark$ in the appropriate cell) whether each of the following sentences is (i) consistent with $\Delta \cup \{\psi\}$ and (ii) logically entailed by $\Delta \cup \{\psi\}$.

<table>
<thead>
<tr>
<th></th>
<th>consistent</th>
<th>logically entailed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(a, a)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r(a, b)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r(b, a)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r(b, b)$</td>
<td></td>
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</tbody>
</table>
Question 4 [10 points]

Consider a language with a single object constant $a$, a single unary function constant $f$, and a relation constants $p$ and $q$.

Suppose that you are given the following premises.

1. $\forall x. (p(x) \rightarrow q(f(x)))$
2. $\forall y. (q(y) \rightarrow p(y)))$
3. $\forall z. p(a)$

Prove $\neg \exists x. (\neg p(x) \land \neg q(x))$ using the Fitch rules of inference and the Induction Axioms. Please annotate your proof by writing the rule of inference and line number(s) for each step in your proof (abbreviations are fine). Clearly mark any assumptions and subproofs within your proof in the same format as the exercises.

Note: Our solution is 20 lines, including the premises.

Hint: This problem requires the use of induction.
Question 5 [10 points]

(a) Convert the following sentence to clausal form. (4 points)

$$\exists x. (\exists y. \forall z. (\neg p(x, y) \lor \neg q(x, y, z)) \rightarrow \exists z. \forall y. q(z, y, x))$$

*Hint*: Our conversion yields 1 clause.
(b) Use resolution to show that the following set of clauses is unsatisfiable.

\[
\begin{align*}
\{ & p(b, X), q(X, Z, Y) \} \\
\{ & \neg p(X, a), q(Y, X, Z) \} \\
\{ & q(b, Y, c), \neg r(Y) \} \\
\{ & \neg q(X, Y, Z), r(Z) \} \\
\{ & \neg q(Y, f(b), c), \neg r(f(Y)) \}
\end{align*}
\]
Fitch Rules of Inference and Induction Axioms

And Introduction
\[ \phi_1 \]
\[ \ldots \]
\[ \phi_n \]
\[ \phi_1 \land \cdots \land \phi_n \]

And Elimination
\[ \phi_1 \land \cdots \land \phi_n \]
\[ \phi_i \]

Or Introduction
\[ \phi_1 \]
\[ \phi_1 \lor \cdots \lor \phi_n \]

Or Elimination
\[ \phi_1 \lor \cdots \lor \phi_n \]
\[ \phi_1 \Rightarrow \psi \]
\[ \ldots \]
\[ \phi_n \Rightarrow \psi \]
\[ \psi \]

Negation Introduction
\[ \phi \Rightarrow \psi \]
\[ \phi \Rightarrow \neg \psi \]
\[ \neg \phi \]

Negation Elimination
\[ \neg \neg \phi \]
\[ \phi \]

Implication Introduction
\[ \phi \vDash \psi \]
\[ \phi \Rightarrow \psi \]

Implication Elimination
\[ \phi \Rightarrow \psi \]
\[ \phi \]
\[ \psi \]

Biconditional Introduction
\[ \phi \Rightarrow \psi \]
\[ \psi \Rightarrow \phi \]
\[ \phi \Leftrightarrow \psi \]

Biconditional Elimination
\[ \phi \Leftrightarrow \psi \]
\[ \phi \Rightarrow \psi \]
\[ \psi \Rightarrow \phi \]
<table>
<thead>
<tr>
<th>Rule Type</th>
<th>Rule</th>
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<tbody>
<tr>
<td><strong>Universal Introduction</strong></td>
<td>$\phi$</td>
</tr>
<tr>
<td></td>
<td>$\forall \nu. \phi$</td>
</tr>
<tr>
<td></td>
<td>where $\nu$ does not appear free in both $\phi$ and an active assumption</td>
</tr>
<tr>
<td><strong>Universal Elimination</strong></td>
<td>$\forall \nu. \phi[\nu]$</td>
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<tr>
<td></td>
<td>$\phi[\tau]$</td>
</tr>
<tr>
<td></td>
<td>where $\tau$ is substitutable for $\nu$ in $\phi$</td>
</tr>
<tr>
<td><strong>Existential Introduction</strong></td>
<td>$\phi[\tau]$</td>
</tr>
<tr>
<td></td>
<td>$\exists \nu. \phi[\nu]$</td>
</tr>
<tr>
<td></td>
<td>where $\tau$ does not contain any universally quantified variables</td>
</tr>
<tr>
<td><strong>Existential Elimination (closed sentences)</strong></td>
<td>$\exists \nu. \phi[\nu]$</td>
</tr>
<tr>
<td></td>
<td>$\phi[[\tau]]$</td>
</tr>
<tr>
<td></td>
<td>where $\tau$ is not an existing object constant</td>
</tr>
<tr>
<td><strong>Existential Elimination (free variables)</strong></td>
<td>$\exists \nu. \phi[\nu_1, \ldots, \nu_n, \nu]$</td>
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<tr>
<td></td>
<td>$\phi[[\pi(\nu_1, \ldots, \nu_n)]]$</td>
</tr>
<tr>
<td></td>
<td>where $\pi$ is not an existing constant</td>
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<tr>
<td><strong>Domain Closure</strong></td>
<td>$\phi[\sigma_1]$</td>
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<tr>
<td></td>
<td>$\ldots$</td>
</tr>
<tr>
<td></td>
<td>$\phi[\sigma_n]$</td>
</tr>
<tr>
<td></td>
<td>$\forall \nu. \phi[\nu]$</td>
</tr>
<tr>
<td><strong>Linear Induction</strong></td>
<td>$\phi[a]$</td>
</tr>
<tr>
<td></td>
<td>$\forall \mu. (\phi[\mu] \rightarrow \phi[s(\mu)])$</td>
</tr>
<tr>
<td></td>
<td>$\forall \nu. \phi[\nu]$</td>
</tr>
<tr>
<td><strong>Structural Induction</strong></td>
<td>$\phi[a]$</td>
</tr>
<tr>
<td></td>
<td>$\phi[b]$</td>
</tr>
<tr>
<td></td>
<td>$\forall \lambda. \forall \mu. ((\phi[\lambda] \land \phi[\mu]) \rightarrow \phi[c(\lambda, \mu)])$</td>
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<tr>
<td></td>
<td>$\forall \nu. \phi[\nu]$</td>
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<tr>
<td><strong>Tree Induction</strong></td>
<td>$\phi[a]$</td>
</tr>
<tr>
<td></td>
<td>$\forall \mu. (\phi[\mu] \rightarrow \phi[f(\mu)])$</td>
</tr>
<tr>
<td></td>
<td>$\forall \mu. (\phi[\mu] \rightarrow \phi[g(\mu)])$</td>
</tr>
<tr>
<td></td>
<td>$\forall \nu. \phi[\nu]$</td>
</tr>
</tbody>
</table>

In addition to these rules of inference, you may make **assumptions** within subproofs and use **reiteration** to reproduce an earlier conclusion in your proof.