CS 157: Final Examination
Fall 2018-19

- Please read all instructions (including these) carefully.
- The exam is closed book, closed notes, and closed internet.
- The exam consists of 9 pages including this page. The last two pages are reference sheets for the Fitch rules of inference and the induction axioms. There are 5 questions. Each question is worth 10 points.
- Time limit: 110 minutes. Budget your time accordingly.
- Please write your solutions in the spaces provided on the exam. Make sure that your solutions are neat and clearly marked. You may use the blank areas and backs of the exam pages for scratch work.

In accordance with both the letter and spirit of the Honor Code, I have neither given nor received assistance on this examination.

NAME: ________________________________
SUNETID (username): ________________________________
SIGNATURE: ________________________________

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Question 1

a. [6 points] State whether the following relational logic sentences are valid, contingent, or unsatisfiable (no explanation is needed):

i. $(\exists x. (\forall y. p(x, y) \Rightarrow p(z, z))) \leftrightarrow (\exists x. p(x, x) \Rightarrow \exists y. p(y, y))$

ii. $(\forall x. (p(x) \lor q(x))) \Rightarrow (\exists y. p(y) \Rightarrow (p(x) \Rightarrow \forall y. p(y)))$

iii. $\exists y. (p(y) \Rightarrow \exists x. q(x, y)) \Rightarrow \neg \exists x. q(y, x)$. 
b. [4 points] Consider a language with object constants \textit{friend}, \textit{foe} and \textit{neutral}; binary function constant \textit{joinforces}; and binary relation constant \textit{defeats}.

Consider the sentences below that axiomatize \textit{defeats}:

\[
\forall x. \forall y. \text{defeats}(\text{joinforces}(x, y), \text{joinforces}(x, \text{neutral}))
\]
\[
\forall x. \forall y. \forall z. (\text{defeats}(\text{joinforces}(x, z), \text{joinforces}(y, z)) \Rightarrow \text{defeats}(x, y))
\]
\[
\text{defeats}(\text{neutral}, \text{friend})
\]
\[
\neg \text{defeats}(\text{friend}, \text{neutral})
\]
\[
\forall x. \neg \text{defeats}(x, x)
\]

Determine which of the following sentences are entailed by the axioms for \textit{defeats}:

\[
\text{defeats}(\text{joinforces}(\text{neutral}, \text{joinforces}(\text{neutral}, \text{foe})), \text{joinforces}(\text{neutral}, \text{friend}))
\]
\[
\text{defeats}(\text{foe}, \text{foe})
\]
\[
\text{defeats}(\text{friend}, \text{joinforces}(\text{neutral}, \text{foe}))
\]
Question 2 [10 points]

The proofs below are incorrect. State the line on which the first error occurs, and briefly explain why the line is incorrect. You may assume c is not an existing object constant.

Part 1 [5 points]

1. \( \vdash p(x) \) (Assumption)
2. \( \vdash \forall x. p(x) \) (UI: 1)
3. \( p(x) \Rightarrow \forall x. p(x) \) (II: 1, 2)
4. \( \forall x. (p(x) \Rightarrow \forall x. p(x)) \) (UI: 3)
5. \( \vdash \exists x. p(x) \) (Assumption)
6. \( \vdash p(c) \) (EE: 5)
7. \( p(c) \Rightarrow \forall x. p(x) \) (UE: 4)
8. \( \vdash \forall x. p(x) \) (IE: 6, 7)
9. \( \exists x. p(x) \Rightarrow \forall x. p(x) \) (II: 5, 8)
Part 2 [5 points]

1. $\forall x. \forall y. (p(x, y) \Rightarrow q(x))$ (Premise)

2. $\vert \exists y. p(x, y)$ (Assumption)

3. $\vert p(x, [c])$ (EE: 2)

4. $\vert \forall y. (p(x, y) \Rightarrow q(x))$ (UE: 1)

5. $\vert p(x, [c]) \Rightarrow q(x)$ (UE: 4)

6. $\vert q(x)$ (IE: 3, 5)

7. $\exists y. p(x, y) \Rightarrow q(x)$ (II: 2, 6)

8. $\forall x. (\exists y. p(x, y) \Rightarrow q(x))$ (UI: 7)
Question 3 [10 points]

Given the premises $\exists x. q(x) \Rightarrow \exists x. p(x)$, $\exists x. \neg r(x)$, and $\forall x. (\neg q(x) \Rightarrow r(x))$, use the Fitch system to prove $\exists x. p(x)$.

Please annotate your proof by writing the rule and line number for each step in your proof (abbreviations are fine). Clearly mark any assumptions and subproofs within your proof in the same format as the exercises and notes.
Question 4 [10 points]

Consider the following Herbrand language:

1. object constants: $a$
2. function constants: $f$ (unary)
3. relation constants: $p$ (unary)

Given the premises

\[
p(a) \\
\forall x. (\neg p(f(x)) \Rightarrow \neg p(x))
\]

use the Fitch system to prove $\forall x. p(x)$. Please annotate your proof by writing the rule and the line number for each step in your proof (abbreviations are fine). Clearly mark any assumptions and subproofs within your proof in the same format as the exercises and notes.
Question 5 [10 points]

a. Convert the following set of two sentences to clausal form. (4 points)

\[ \forall x. (p(x) \lor p(f(x))) \Rightarrow \forall x. q(x, f(x)) \]
\[ \forall x. \exists y. q(x, y) \]
b. Use resolution to show that the following set of clauses is unsatisfiable.

Please annotate your proof by writing the rule and line number for each step in your proof (abbreviations are fine). (6 points)

1. \{\neg r(x)\}
2. \{r(x), p(x)\}
3. \{\neg p(a), q(x), q(c)\}
4. \{\neg p(b), \neg q(x)\}
## Fitch Rules of Inference and Induction Axioms

### And Introduction
\[
\phi_1, \ldots, \phi_n \\
\hline
\phi_1 \land \cdots \land \phi_n
\]

### And Elimination
\[
\phi_1 \land \cdots \land \phi_n \\
\hline
\phi_i
\]

### Or Introduction
\[
\phi_1 \\
\hline
\phi_1 \lor \cdots \lor \phi_n
\]

### Or Elimination
\[
\phi_1 \lor \cdots \lor \phi_n \\
\phi_1 \Rightarrow \psi \quad \ldots \quad \phi_n \Rightarrow \psi \\
\hline
\psi
\]

### Negation Introduction
\[
\phi \Rightarrow \psi \\
\phi \Rightarrow \neg \psi \\
\hline
\neg \phi
\]

### Negation Elimination
\[
\neg \neg \phi \\
\hline
\phi
\]

### Implication Introduction
\[
\phi \vdash \psi \\
\hline
\phi \Rightarrow \psi
\]

### Implication Elimination
\[
\phi \Rightarrow \psi \\
\phi \\
\hline
\psi
\]

### Biconditional Introduction
\[
\phi \Rightarrow \psi \\
\psi \Rightarrow \phi \\
\hline
\phi \Leftrightarrow \psi
\]

### Biconditional Elimination
\[
\phi \Leftrightarrow \psi \\
\hline
\phi \Rightarrow \psi \\
\psi \Rightarrow \phi
\]
Universal Introduction
\[ \phi \]
\[ \forall \nu. \phi \]
where \( \nu \) does not appear free in both \( \phi \) and an active assumption

Universal Elimination
\[ \forall \nu. \phi[\nu] \]
\[ \phi[\tau] \]
where \( \tau \) is substitutable for \( \nu \) in \( \phi \)

Existential Introduction
\[ \phi[\tau] \]
\[ \exists \nu. \phi[\nu] \]
where \( \tau \) does not contain any universally quantified variables

Existential Elimination (closed sentences)
\[ \exists \nu. \phi[\nu] \]
\[ \phi[[\tau]] \]
where \( \tau \) is not an existing object constant

Existential Elimination (free variables)
\[ \exists \nu. \phi[\nu_1, \ldots, \nu_n, \nu] \]
\[ \phi[[\pi(\nu_1, \ldots, \nu_n)]] \]
where \( \pi \) is not an existing constant

Domain Closure
\[ \phi[\sigma_1] \]
\[ \ldots \]
\[ \phi[\sigma_n] \]
\[ \forall \nu. \phi[\nu] \]

Linear Induction
\[ \phi[a] \]
\[ \forall \mu. (\phi[\mu] \rightarrow \phi[s(\mu)]) \]
\[ \forall \nu. \phi[\nu] \]

Tree Induction
\[ \phi[a] \]
\[ \forall \mu. (\phi[\mu] \rightarrow \phi[f(\mu)]) \]
\[ \forall \mu. (\phi[\mu] \rightarrow \phi[g(\mu)]) \]
\[ \forall \nu. \phi[\nu] \]

Structural Induction
\[ \phi[a] \]
\[ \phi[b] \]
\[ \forall \lambda. \forall \mu. ((\phi[\lambda] \land \phi[\mu]) \rightarrow \phi[c(\lambda, \mu)]) \]
\[ \forall \nu. \phi[\nu] \]

In addition to these rules of inference, you may make assumptions within subproofs and use reiteration to reproduce an earlier conclusion in your proof.