CS 157: Final Examination  
Fall 2017-18

- Please read all instructions (including these) carefully.
- The exam is closed book, closed notes, and closed internet.
- The exam consists of 9 pages including this page. The last two pages are reference sheets for the Fitch rules of inference and the induction axioms. There are 5 questions. Each question is worth 10 points.
- Time limit: 110 minutes. Budget your time accordingly.
- Please write your solutions in the spaces provided on the exam. Make sure that your solutions are neat and clearly marked. You may use the blank areas and backs of the exam pages for scratch work.

In accordance with both the letter and spirit of the Honor Code, I have neither given nor received assistance on this examination.

NAME:  
SUNETID (username):  
SIGNATURE:

<table>
<thead>
<tr>
<th>Question 1 (10 points)</th>
<th>Question 2 (10 points)</th>
<th>Question 3 (10 points)</th>
<th>Question 4 (10 points)</th>
<th>Question 5 (10 points)</th>
<th>Total Score (50 points)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Question 1

Consider a language with object constants $a$ and $b$ and relation constants $p$ and $q$, where $p$ has arity 1 and $q$ has arity 2. The following is a truth assignment for this language.

\[
\begin{align*}
    p(a)^i &= 0 \\
    p(b)^i &= 0 \\
    q(a, a)^i &= 1 \\
    q(a, b)^i &= 0 \\
    q(b, a)^i &= 0 \\
    q(b, b)^i &= 1
\end{align*}
\]

Indicate whether each of the following relational logic sentences below is true or false under this truth assignment. No justification is required.

a. $\exists x. (p(x) \lor \forall y. q(y, x))$ (2 points)

   TRUE \hspace{2cm} FALSE

b. $\forall x. \forall y. (q(x, y) \Rightarrow p(y))$ (2 points)

   TRUE \hspace{2cm} FALSE

c. $\neg \exists y. \forall x. q(x, y)$ (2 points)

   TRUE \hspace{2cm} FALSE

d. $\forall x. \exists y. q(x, y)$ (2 points)

   TRUE \hspace{2cm} FALSE

e. $\neg \exists x. \neg p(x)$ (2 points)

   TRUE \hspace{2cm} FALSE
Question 2 [10 points]

Indicate whether each of the relational logic sentences below is valid, contingent, or unsatisfiable by circling the appropriate term. No justification is required.

a. $\exists x. (p(x) \Rightarrow \forall y. p(y))$ (2.5 points)

   VALID   CONTINGENT   UNSATISFIABLE

b. $\forall x. p(x) \land (\forall x. (p(x) \Rightarrow q(s(x)))) \Rightarrow \forall x.q(x))$ (2.5 points)

   VALID   CONTINGENT   UNSATISFIABLE

c. $\neg \exists x. (p(x) \land r(x)) \Rightarrow \forall x. (\neg p(x) \lor \neg r(x))$ (2.5 points)

   VALID   CONTINGENT   UNSATISFIABLE

d. $\forall x. (p(x) \lor q(x)) \land \forall x. (p(x) \Rightarrow q(x)) \land \exists x. \neg q(x)$ (2.5 points)

   VALID   CONTINGENT   UNSATISFIABLE
Question 3 [10 points]

Given the premise \( \neg \forall x. p(x) \), use the Fitch system to prove \( \exists x. \neg p(x) \).

Please annotate your proof by writing the rule and line number for each step in your proof (abbreviations are fine). Clearly mark any assumptions and subproofs within your proof in the same format as the exercises and notes.
Question 4 [10 points]

Consider the following Herbrand language:

1. object constants: $a$
2. function constants: $f$ (unary)
3. relation constants: $p$ (unary), $q$ (unary)

Given the premises

$$
p(a)
\forall x. (p(x) \Rightarrow q(f(x)))
\forall x. (q(x) \Rightarrow p(f(x))),
$$

use the Fitch system to prove $\forall x. (p(x) \lor q(x))$. Please annotate your proof by writing the rule and line number for each step in your proof (abbreviations are fine). Clearly mark any assumptions and subproofs within your proof in the same format as the exercises and notes. For reference, our solution takes 19 lines, including the premises.
Question 5 [10 points]

a. Convert the following sentence to clausal form. (3 points)

\[ \neg \forall x. (p(x) \Rightarrow \exists y. (r(y) \land \forall y. q(x, y))) \]

*Hint:* Our conversion yields 2 clauses.
b. Use resolution to show that the following set of clauses is unsatisfiable.

Please annotate your proof by writing the rule and line number for each step in your proof (abbreviations are fine). (7 points)

1. \{\neg p(y, x), q(y, b)\}
2. \{p(x, y), q(a, y)\}
3. \{r(a, a), \neg q(a, b)\}
4. \{\neg r(a, a), p(x, x)\}
5. \{\neg p(x, x), \neg r(a, x)\}
Fitch Rules of Inference and Induction Axioms

And Introduction

\[ \phi_1 \]

\[ \ldots \]

\[ \phi_n \]

\[ \phi_1 \land \ldots \land \phi_n \]

And Elimination

\[ \phi_1 \land \ldots \land \phi_n \]

\[ \underline{\phi_i} \]

Or Introduction

\[ \phi_1 \]

\[ \phi_1 \lor \ldots \lor \phi_n \]

Or Elimination

\[ \phi_1 \lor \ldots \lor \phi_n \]

\[ \phi_1 \Rightarrow \psi \]

\[ \ldots \]

\[ \phi_n \Rightarrow \psi \]

\[ \underline{\psi} \]

Negation Introduction

\[ \phi \Rightarrow \psi \]

\[ \phi \Rightarrow \neg\psi \]

\[ \neg\phi \]

Negation Elimination

\[ \neg\neg\phi \]

\[ \underline{\phi} \]

Implication Introduction

\[ \phi \vdash \psi \]

\[ \underline{\phi \Rightarrow \psi} \]

Implication Elimination

\[ \phi \Rightarrow \psi \]

\[ \phi \]

\[ \underline{\psi} \]

Biconditional Introduction

\[ \phi \Rightarrow \psi \]

\[ \psi \Rightarrow \phi \]

\[ \phi \leftrightarrow \psi \]

Biconditional Elimination

\[ \phi \leftrightarrow \psi \]

\[ \phi \Rightarrow \psi \]

\[ \psi \Rightarrow \phi \]
Universal Introduction

\[ \phi \]

\[ \forall \nu . \phi \]

where \( \nu \) does not appear free in both \( \phi \) and an active assumption

Universal Elimination

\[ \forall \nu . \phi[\nu] \]

\[ \phi[\tau] \]

where \( \tau \) is substitutable for \( \nu \) in \( \phi \)

Existential Introduction

\[ \phi[\tau] \]

\[ \exists \nu . \phi[\nu] \]

Existential Elimination (closed sentences)

\[ \exists \nu . \phi[\nu] \]

\[ \phi[\{\tau\}] \]

where \( \tau \) is not an existing object constant

Existential Elimination (free variables)

\[ \exists \nu . \phi[\nu_1, \ldots, \nu_n, \nu] \]

\[ \phi[\{\pi(\nu_1, \ldots, \nu_n)\}] \]

where \( \pi \) is not an existing constant

Domain Closure

\[ \phi[\sigma_1] \]

\[ \ldots \]

\[ \phi[\sigma_n] \]

\[ \forall \nu . \phi[\nu] \]

Linear Induction

\[ \phi[a] \]

\[ \forall \mu . (\phi[\mu] \rightarrow \phi[s(\mu)]) \]

\[ \forall \nu . \phi[\nu] \]

Tree Induction

\[ \phi[a] \]

\[ \forall \mu . (\phi[\mu] \rightarrow \phi[f(\mu)]) \]

\[ \forall \mu . (\phi[\mu] \rightarrow \phi[g(\mu)]) \]

\[ \forall \nu . \phi[\nu] \]

Structural Induction

\[ \phi[a] \]

\[ \phi[b] \]

\[ \forall \lambda \forall \mu . ((\phi[\lambda] \land \phi[\mu]) \rightarrow \phi[c(\lambda, \mu)]) \]

\[ \forall \nu . \phi[\nu] \]

In addition to these rules of inference, you may make **assumptions** within subproofs and use **reiteration** to reproduce an earlier conclusion in your proof.